Few aspects of the electroweak interaction

Exercise No. 1: Experimental consequences of the V-A interaction (3 points)

The V-A structure of the weak interaction is a crucial feature of the Standard Model, both for the construction of the theory and from a phenomenology point of view. This exercise aims to address a couple of consequences of its parity-violating structure.

1. The charged pion $\pi^-$ decays almost exclusively into a $(\mu^-, \bar{\nu}_\mu)$ pair compared to its electronic decay:

$$ R \equiv \frac{\text{BR}(\pi^- \to e^- \bar{\nu}_e)}{\text{BR}(\pi^- \to \mu^- \bar{\nu}_\mu)} \sim 10^{-4} \quad (1) $$

However, the available phase space of a decay into a $(e^-, \bar{\nu}_e)$ pair is much larger since $m_e = 0.5 \text{ MeV}$, $m_\mu = 106 \text{ MeV}$ and $m_\pi = 140 \text{ MeV}$ and (2) the weak interaction has the same strength for each flavour. Explain this apparent paradox with qualitative arguments. In particular, identify the role of the non-negligible mass of the muon, compared to the mass of the pion. (Hint: how are helicity and chirality related?)

2. Let us consider a $\tau$ lepton produced with a momentum $\vec{p}$ and decaying into a $(\pi^-, \nu_\tau)$ pair. Explain how the helicity the $\tau$ can modify the momentum of the pion measured in the detector.

3. Let’s consider a spin-0 particle decaying into a $WW$ pair. Considering the di-leptonic final state of the $WW$ pair i.e. $WW \to \ell \nu \ell \nu$, what can you say about the angle between the two leptons?

Exercise No. 2: Muon decay (7 points)

In this exercise, we want to compute the lifetime of the muon using the process

$$ \mu^-(p) \to e^-(p') \bar{\nu}_e(k') \nu_\mu(k) \quad (2) $$

1. Draw the Feynman diagram of the process in the Fermi theory of the weak interaction. Following the structure “current $\times$ propagator $\times$ current”, write the associated invariant amplitude $i \cdot \mathcal{M}$, as a function of momentum and the Fermi constant $G_F$.

2. Show that the squared amplitude is given by

$$ |\mathcal{M}|^2 = \frac{G_F^2}{2} \left[ \bar{u}(k)\gamma^\mu(1 - \gamma^5)u(p) \bar{u}(p')\gamma_\mu(1 - \gamma^5)v(k') \right] \times \left[ \bar{v}(k')(1 - \gamma^5)\gamma_\nu u(p') \bar{u}(p)\gamma_\nu(1 - \gamma^5)u(k) \right] \quad (3) $$
3. By averaging over the spin configuration of the initial muon, by summing over the spin configuration of the final particles, and by neglecting the mass of the electron, show that

\[ |\mathcal{M}|^2 = 64 \ G_F^2 \ (k \cdot p') \ (k' \cdot p) \]  

(4)

Hint: we can use the following formula (where \( p^\mu = \gamma^\mu p \))

\[
\text{Tr} \left[ \gamma^\mu (1 - \gamma^5) \gamma^\nu (1 - \gamma^5) \gamma^\alpha (1 - \gamma^5) \gamma^\beta \right] \text{Tr} \left[ \gamma^\mu (1 - \gamma^5) \gamma^\nu (1 - \gamma^5) \gamma^\alpha (1 - \gamma^5) \gamma^\beta \right] = 256 \ (p_1 \cdot p_3) (p_2 \cdot p_4)
\]  

(5)

4. Considering a muon at rest, \( p = (m_\mu, \vec{0}) \), show that

\[ |\mathcal{M}|^2 = 64 \ G_F^2 \ (m_\mu^2 - 2m_\mu E_{\bar{\nu}_e}) \ m_\mu E_{\bar{\nu}_e} \]  

(6)

5. The decay rate \( d\Gamma_\mu \) by unit of phase space volume is given by the Golden Fermi rule:

\[ d\Gamma_\mu = \frac{1}{2m_\mu} \frac{|\mathcal{M}|^2}{dQ_3} \]  

(7)

where \( dQ_3 \) is the three-body phase space

\[
dQ_3 = \frac{dp'}{2\pi^2} \frac{dk'}{2\pi^2} \frac{dk}{2\pi^2} (2\pi)^4 \delta^4(p - p' - k' - k)
\]  

(8)

- Integrate over \( k \).
- We will now integrate over \( \vec{k} \) in several steps. By using \( |\vec{p}_\nu| = E_\nu \) for each neutrino, first show that

\[
d\Gamma_\mu = \left( \int_{E_{\bar{\nu}_e}, E_e} \frac{E_{\bar{\nu}_e} \sin \theta dE_{\bar{\nu}_e} d\theta d\phi}{E_\nu} \delta(m_\mu - E_{\bar{\nu}_e} - E_\nu - E_e) \right) \frac{|\mathcal{M}|^2}{8(2\pi)^3 m_\mu^2} \frac{dp'}{2E_e}
\]  

After having integrated over \( \phi \), perform an integration variable change

\[
u = \sqrt{E_{\bar{\nu}_e}^2 + E_e^2 + 2E_{\bar{\nu}_e}E_e \cos \theta}
\]  

(10)

and show that

\[
d\Gamma_\mu = \left( \int_{|E_{\bar{\nu}_e} + E_e|}^{E_{\bar{\nu}_e} + E_e} \delta(m_\mu - E_{\bar{\nu}_e} - u - E_e) du \right) \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_\mu} \frac{dp'}{E_e}
\]  

(11)

Finally, by analyzing when the \( \delta \) function in equation (11) is not zero, show that

\[
d\Gamma_\mu = \left( \int_{\frac{1}{2}m_\mu - E_e}^{\frac{1}{2}m_\mu - E_e} \frac{|\mathcal{M}|^2}{16(2\pi)^4 m_\mu} dE_{\bar{\nu}_e} \right) \frac{dp'}{E_e}
\]  

(12)

6. Using equations (6) and (12), show that the decay rate per unit of energy of the emitted electron is

\[
\frac{d\Gamma_\mu}{dE_e} = \frac{32 \ G_F^2 \ m_\mu E_e^2}{(4\pi)^3} \left( \frac{m_\mu}{2} - \frac{2}{3} E_e \right)
\]  

(13)

Hint: convert the integral over \( p' \) into an integral over \( E_e \). Plot this function and give the most likely energy of the emitted electron.

7. Compute the total lifetime of the muon. Experimental measurements give \( m_\mu = 105.65 \text{ MeV} \) and \( \tau_{\text{life}} = 2.20 \times 10^{-6} \text{ s} \).
• Deduce the value of the Fermi constant.

• In a historical perspective, let’s assume that an underlying dynamic is responsible of this decay. Write $G_F$ as a function of the mass of the new force mediator $m_X$ and its coupling constant $g_X$.

• Assuming the same strength that the electromagnetic interaction ($g_X = g_{EM}$), compute the mass of the new force mediator.

• Given the mass of the $W$ boson, deduce the value of $\alpha_{\text{weak}} = g_W^2/4\pi$. Compare with $\alpha_{EM}$ and comment on the “weakness” of the weak interaction.