Exercises for Advanced Particle Physics - Winter term 2013/14
Exercise sheet No. I

Prof. Karl Jakobs, Dr. Romain Madar, Claudia Giuliani

The solutions have to be returned to mail box no. 1
in the foyer of the Gustav-Mie-House before **Monday, November 4th, 12:00h**.

---

1 Relativistic kinematics (4 points)

Following the two postulates of the relativity, the coordinate transformation between two frames lead to non intuitive effects (e.g. the time becomes frame-dependent, just as the position in space). The next exercises try to recap the basic description of these effects and their pratical consequences.

Exercise No. 1 **(2 points)**

We consider two frames $\mathcal{R}$ and $\mathcal{R}'$, with a relative velocity $v(\mathcal{R}'/\mathcal{R}) \equiv v$ along the Oz direction. We denote (x, y, z, t) (resp. (x', y', z', t')) the coordinates of a space-time point in $\mathcal{R}$ (resp. $\mathcal{R}'$):

(i) Write down the Lorentz transformation linking the coodinates in the two frames, using $\beta = v/c$ and $\gamma = 1/\sqrt{1-\beta^2}$. Check that the quantity $\Delta s^2 = (c\Delta t)^2 - (\Delta \mathbf{r})^2$ is invariant (frame-independent).

(ii) Consider an excited nucleus (with a life time τ_{life}) moving at a velocity v in the lab frame, compute its decay time and the traveled distance before it decays, measured in the lab. By considering an ideal clock made of light and mirrors moving at a velocity v, try to *explain* this result. Numerical examples:

- At which velocity should we move to see an apple falling down from 1m in 1h ?
- What would be the traveled distance of a muon moving at $0.999\,c$, without any relativistic effects ($\tau_{\text{life}} = 2.2\ \mu$s) ? Compare with the relativistic theory prediction.

(iii) We describe a wave propagation by its frequency ω and its wave vector $\mathbf{k}$. $(\omega, c\mathbf{k})$ form a 4-vector with $\omega^2 = (c\mathbf{k})^2$. Let's assume we have light source, moving at a velocity v which emits two plane waves having a angle $+\theta$ and $-\theta$ with the Oz axis. Compute the angle between the 2 plane waves in the lab frame. Discuss the result. Assuming this phenomenon is general, what practical consequences could you see?

Exercise No. 2 **(2 points)**

Kinematic of basic reactions:

(i) Express the energy-momentum 4-vector of particles with a moving mass m_0 at a velocity v, in terms of γ, m_0 and v. Compute the norm of this 4-vector. What is the relation between energy, momentum and mass? Study the limit for an ultra-relativistic particle ($v/c \approx 1$). Interpret the Taylor developement of the energy in the non-relativistic limit ($v/c \ll 1$).

(ii) By considering a symmetric collision between two electrons, compute the energy, the momentum and the velocity of each electron to be able to produce the Higgs boson ($m_H = 125$ GeV$/c^2$). Compute the electron energy for a fixed target collision (only one electron is moving, the other one is contained in a target). Discuss the result.

(iii) Consider a particle A with a mass M at rest, decaying into two particles b and c of masses m_1 and m_2. Compute the energy and the momentum of the particle b and c. Study the behavior of the decay kinematic as a function of m/M (for $m \equiv m_1 = m_2$).

2 Neutrino masses and oscillations (6 points)

The neutrinos, introduced to save the energy conservation in β-decay, were assumed to be massless in agreement with the available observations. Since the end of the 90's, flavour oscillations of neutrinos have been experimentally confirmed and have led to attribute non-zero masses to neutrinos. These exercises try to illustrate how neutrino masses can be probed experimentally.

Exercise No. 3 **(3 points)**

We consider two neutrinos states which can be written in two different bases:

(i) the interaction (or flavour) basis $\{|\nu_e\rangle, |\nu_\mu\rangle\}$. All kind of interactions between neutrinos and matter (production, detection) can only happen for the flavour eigenstates.

(ii) the Hamiltonian (or mass) basis $\{|\nu_1\rangle, |\nu_2\rangle\}$. Free evolution will be stationary only for the Hamiltonian eigenstates.

Without loss of generality, we can write the transformation between the two basis using one parameter θ, called mixing angle :

$$|\nu_e\rangle \;=\; \cos\theta\,|\nu_1\rangle \;+\; \sin\theta\,|\nu_2\rangle \quad ; \quad |\nu_\mu\rangle \;=\; -\sin\theta\,|\nu_1\rangle \;+\; \cos\theta\,|\nu_2\rangle \tag{1}$$

1. Recall the time evolution of Hamiltonian eigenstate $|\psi_i\rangle$ defined by $\hat{H}\psi_i = E_i\psi_i$. Considering the free evolution of a neutrino having a momentum $\mathbf{p}$ and a mass m_i, write the energy E_i as a function of $\mathbf{p}$, m_i and c. Simplify this expression by considering the mass of a neutrino to be small compared to its kinetic energy (but not negligible).

2. Compute the state vector of the system $|\psi(t)\rangle$ at anytime t, given the initial state $|\psi(0)\rangle = |\nu_e\rangle$. Show that the probability $\mathcal{P}_{\nu_e\to\nu_e}(t)$ to measure an electron neutrino at a time t is :

$$\mathcal{P}_{\nu_e\to\nu_e}(t) \;=\; 1 - \sin^2(2\theta)\sin^2\left(\frac{\pi c t}{L}\right) \quad ; \quad L = \frac{4\pi\hbar p}{|\Delta m^2|c^2} \;\;,\;\; \Delta m^2 \equiv m_2^2 - m_1^2 \tag{2}$$

   Plot $\mathcal{P}_{\nu_e\to\nu_e}(t)$ and explain when the oscillations are maximal and when they are minimal.

3. Do you know a way to experimentally produce electronic neutrinos on earth? Assuming we are able to measure the neutrino flux with a precision of $\pm5\%$, what is the minimal distance from which you could see these oscillations ($\theta = 45°$, $|\Delta m^2|c^4 = 10^{-4}$ eV^2, $pc = 4$ MeV) ? How does this distance change if the mixing angle is not maximal (i.e. $\theta \neq 45°$) ?

Exercise No. 4 **(3 points)**

Neutrino oscillation experiments are only able to measure neutrino mass *differences* but cannot provide any information on the *absolute mass scale* of these particles. One possible way of measuring the absolute neutrino mass is to experimentally study the β-decay kinematic :

$$n \;\to\; p + e^- + \bar{\nu}_e \tag{3}$$

1. Compute the electron energy (E_e) as a function of the proton mass (m_p) and the invariant mass of the $\{p, \bar{\nu}_e\}$ system ($m_{p\bar{\nu}_e}$).

2. In which kinematic configuration E_e is minimal? What is the value of $E_e^{\min}$? In which kinematic configuration E_e is maximal? What is the value of $E_e^{\max}$?

3. The Karlsruhe Tritium Neutrino Experiment (KATRIN) tries to measure the highest energetic electrons coming from β-decays of tritium nuclei. Using the previous question, explain how this measurement can provide information on the absolute neutrino mass scale. Interpret with simple kinematic arguments. Based on the reaction (3), find out what would be the maximal measured energy of the electron to probe a neutrino mass of 0.05 eV (we could, for example, plot $E_e^{\max}$ versus m_ν)? Use $m_p\, c^2 = 938.272$ MeV, $m_n\, c^2 = 939.565$ MeV, $m_e\, c^2 = 0.511$ MeV.