Exercises for Advanced Particle Physics - Winter term 2013/14
Exercise sheet No. IV

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The solutions have to be returned to mail box no. 1
in the foyer of the Gustav-Mie-House before **Monday, November 25th, 12:00h**.

Elementary processes, transition amplitudes and cross sections

During the previous lectures, the propagation of free particles was described in a relativistic framework. To better understand particle physics, we want to learn about how particles *interact*. Experimentally, we analyse the final state of collisions produced with a known initial state. These exercises aim to illustrate how the properties of the final state can be predicted by a given theory.

Exercise No. 1: Generic form of the cross section **(1+2 points)**

Based on Fermi's Golden Rule, the generic form of the cross section can be written, as seen in the lecture:

$$\mathrm{d}\sigma \;=\; \frac{1}{F}\,|\mathcal{M}|^2 \mathrm{d}Q \tag{1}$$

1. Discuss the physical meaning of each term in this equation.

2. *Optional question (2 points).* For the reaction $a + b \to c + d$ with a total energy E_{cm}, show that the differential cross section in the centre-of-mass frame ($\vec{p}_a + \vec{p}_b = \vec{0}$) can be written

$$\frac{\mathrm{d}\sigma}{\mathrm{d}\Omega} \;=\; \frac{|\mathcal{M}|^2}{64\pi^2 E_{\mathrm{cm}}^2}\frac{|\vec{p}_c|}{|\vec{p}_a|} \tag{2}$$

   Hint: Use the following relations:

$$F = 4\,\sqrt{(p_a\cdot p_b)^2 - m_a^2 m_b^2} \tag{3}$$

$$\mathrm{d}Q \;=\; \frac{\mathrm{d}\vec{p}_c}{(2\pi)^3 2E_c}\,\frac{\mathrm{d}\vec{p}_d}{(2\pi)^3 2E_d}\,(2\pi)^4\delta^4(p_a + p_b - p_c - p_d) \tag{4}$$

   and the following transformation of the integration variable

$$p \;\to\; \sqrt{m_c^2 + p^2} + \sqrt{m_d^2 + p^2} \tag{5}$$

Exercise No. 2: Creation of muon pairs in e^+e^- collisions in QED **(9 points)**

The goal of this exercise is to calculate and interpret the unpolarized differential and total cross section of the following process predicted by quantum electrodyamics.

$$e^+(p_1)\,e^-(p_2) \;\to\; \mu^+(p_3)\,\mu^-(p_4) \tag{6}$$

(p_1, p_2, p_3 and p_4 denote the four momenta of the particles).

1. Draw the Feynman diagram of the process. Following the structure "current × propagator × current", write the associated invariant amplitude $i\cdot\mathcal{M}$ (note $q \equiv p_2 + p_1$).

2. Show that the squared amplitude is given by

$$|\mathcal{M}|^2 = \frac{e^4}{q^4}\left[\bar{v}(p_1)\gamma^\mu u(p_2)\bar{u}(p_2)\gamma^\nu v(p_1)\right]\left[\bar{u}(p_4)\gamma_\mu v(p_3)\bar{v}(p_3)\gamma_\nu u(p_4)\right] \tag{7}$$

3. We assume that the experimental setup is neither able to produce polarized beams nor able to measure the spin state of the out-coming particles. Explain why the expression of the matrix element involved in equation (1) is

$$\overline{|\mathcal{M}|^2} \equiv \frac{1}{4}\sum_{\text{spins}} |\mathcal{M}|^2 \tag{8}$$

4. Show that

$$\overline{|\mathcal{M}|^2} = \frac{e^4}{4q^4}\text{tr}\left[(\not{p}_1 - m_e)\gamma^\mu(\not{p}_2 + m_e)\gamma^\nu\right]\text{tr}\left[(\not{p}_4 + m_\mu)\gamma_\mu(\not{p}_3 - m_\mu)\gamma_\nu\right] \tag{9}$$

Hint: Write in spinor indices, and use the completeness relations $\sum_s u^s(p)\bar{u}^s(p) = \not{p} + m$ and $\sum_s v^s(p)\bar{v}^s(p) = \not{p} - m$.

5. Using the approximation $m_e \to 0$ and the theorems for traces involving γ matrices from the lecture, show that this expression can be simplified to

$$\overline{|\mathcal{M}|^2} = \frac{8e^4}{q^4}\left[(p_2 \cdot p_4)(p_1 \cdot p_3) + (p_2 \cdot p_3)(p_1 \cdot p_4) + m_\mu^2(p_1 \cdot p_2)\right] \tag{10}$$

6. Considering that the e^+e^- collision are symmetric ($\vec{p}_1 = -\vec{p}_2$), show that $\overline{|\mathcal{M}|^2}$ can be simplified to

$$\overline{|\mathcal{M}|^2} = e^4\left[1 + \frac{m_\mu^2}{E^2} + (1 - \frac{m_\mu^2}{E^2})\cos^2\theta\right] \tag{11}$$

where θ is the angle between p_1 and p_3, E is the energy of any of the four fermions.

7. Using equation (2), show that the differential cross section and the total cross section are given by ($\alpha \equiv e^2/(4\pi)$)

$$\frac{d\sigma}{d\Omega} = \frac{\alpha^2}{16E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left[1 + \frac{m_\mu^2}{E^2} + (1 - \frac{m_\mu^2}{E^2})\cos^2\theta\right] \tag{12}$$

$$\sigma_{\text{total}} = \frac{\pi\alpha^2}{3E^2}\sqrt{1 - \frac{m_\mu^2}{E^2}}\left(1 + \frac{m_\mu^2}{2E^2}\right) \tag{13}$$

8. Discussion of the result.
   - What minimum energy of the incoming fermions is required for this result to be valid?
   - What would be $\sigma(E)$ if $\mathcal{M}$ were constant ? Compare with the behaviour of $\sigma(E)$ defined in equation (13) and deduce an experimental way to probe QED interactions.
   - Express these cross sections in terms of the centre-of-mass energy $\sqrt{s}$ using the high-energy approximation $E \gg m_\mu$. Interprete the result.