Exercises for Advanced Particle Physics - Winter term 2013/14
Exercise sheet No. IX

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The solutions have to be returned to mail box no. 1
in the foyer of the Gustav-Mie-House before **Monday, January 20th, 12:00h**.

Few aspects of the electroweak interaction

Exercise No. 1: Experimental consequences of the V-A interaction (3 points)

The V-A structure of the weak interaction is a crucial feature of the Standard Model, both for the construction of the theory and from a phenomenology point of view. This exercise aims to address a couple of consequences of its parity-violating structure.

1. The charged pion π^- decays almost exclusively into a $(\mu^-, \bar{\nu}_\mu)$ pair compared to its electronic decay:
$$R \equiv \frac{\mathcal{BR}(\pi^- \to e^- \bar{\nu}_e)}{\mathcal{BR}(\pi^- \to \mu^- \bar{\nu}_\mu)} \sim 10^{-4} \quad (1)$$
However, the available phase space of a decay into a $(e^-, \bar{\nu}_e)$ pair is much larger since $m_e = 0.5$ MeV, $m_\mu = 106$ MeV and $m_\pi = 140$ MeV and (2) the weak interaction has the same strength for each flavour. Explain this apparent paradox with qualitative arguments. In particular, identify the role of the non-negligible mass of the muon, compared to the mass the pion. (Hint: how are helicity and chirality related?)

2. Let us consider a τ lepton produced with a momentum $\vec{p}$ and decaying into a (π^-, ν_τ) pair. Explain how the helicity the τ can modify the momentum of the pion measured in the detector.

3. Let's consider a spin-0 particule decaying into a WW pair. Considering the di-leptonic final state of the WW pair *i.e.* $WW \to \ell\nu\ell\nu$, what can you say about the angle between the two leptons?

Exercise No. 2: Muon decay (7 points)

In this exercise, we want to compute the lifetime of the muon using the process
$$\mu^-(p) \;\to\; e^-(p')\; \bar{\nu}_e(k')\; \nu_\mu(k) \quad (2)$$

1. Draw the Feynman diagram of the process in the Fermi theory of the weak interaction. Following the structure "current × propagator × current", write the associated invariant amplitude $i \cdot \mathcal{M}$, as a function of momentum and the Fermi constant G_F.

2. Show that the squared amplitude is given by
$$\begin{aligned} |\mathcal{M}|^2 \;=\; & \frac{G_F^2}{2} \left[\bar{u}(k)\gamma^\mu(1-\gamma^5)u(p)\; \bar{u}(p')\gamma_\mu(1-\gamma^5)v(k')\right] \\ & \times \left[\bar{v}(k')(1-\gamma^5)\gamma_\nu u(p')\; \bar{u}(p)\gamma_\nu(1-\gamma^5)u(k)\right] \end{aligned} \quad (3)$$

3. By averaging over the spin configuration of the intial muon, by summing over the spin configuration of the final particles, and by neglecting the mass of the electron, show that

$$\overline{|\mathcal{M}|^2} \; = \; 64\, G_F^2\, (k \cdot p')\, (k' \cdot p) \tag{4}$$

Hint: we can use the following formula (where $\not{p} \equiv \gamma_\mu p^\mu$)

$$\mathrm{Tr}\left[\gamma^\mu (1-\gamma^5) \not{p}_1 \gamma^\nu (1-\gamma^5) \not{p}_2\right] \mathrm{Tr}\left[\gamma_\mu (1-\gamma^5) \not{p}_3 \gamma_\nu (1-\gamma^5) \not{p}_4\right] = 256\, (p_1 \cdot p_3)(p_2 \cdot p_4) \tag{5}$$

4. Considering a muon at rest, $p = (m_\mu, \vec{0})$, show that

$$\overline{|\mathcal{M}|^2} \; = \; 64\, G_F^2\, (m_\mu^2 - 2 m_\mu\, E_{\bar{\nu}_e})\, m_\mu E_{\bar{\nu}_e} \tag{6}$$

5. The decay rate $\mathrm{d}\Gamma_\mu$ by unit of phase space volume is given by the Golden Fermi rule:

$$\mathrm{d}\Gamma_\mu \; = \; \frac{1}{2 m_\mu}\, \overline{|\mathcal{M}|^2}\, \mathrm{d}Q_3 \tag{7}$$

where $\mathrm{d}Q_3$ is the three-body phase space

$$\mathrm{d}Q_3 \; = \; \frac{\mathrm{d}\vec{p'}}{(2\pi)^3 2E_e}\, \frac{\mathrm{d}\vec{k'}}{(2\pi)^3 2E_{\bar{\nu}_e}}\, \frac{\mathrm{d}\vec{k}}{(2\pi)^3 2E_{\nu_\mu}}\, (2\pi)^4 \delta^4(p - p' - k' - k) \tag{8}$$

   - Integrate over $\vec{k}$.
   - We will now integrate over $\vec{k'}$ in several steps. By using $|\vec{p}_\nu| = E_\nu$ for each neutrino, first show that

$$\mathrm{d}\Gamma_\mu \; = \; \left(\int_{\theta,\phi,E_{\bar{\nu}_e}} \frac{E_{\bar{\nu}_e} \sin\theta \mathrm{d}E_{\bar{\nu}_e} \mathrm{d}\theta \mathrm{d}\phi}{E_{\nu_\mu}}\, \delta(m_\mu - E_{\bar{\nu}_e} - E_{\nu_\mu} - E_e) \right) \frac{\overline{|\mathcal{M}|^2}}{8(2\pi)^5 m_\mu} \frac{\mathrm{d}\vec{p'}}{2E_e} \tag{9}$$

   After having integrated over ϕ, perform an integration variable change

$$u = \sqrt{E_{\bar{\nu}_e}^2 + E_e^2 + 2 E_{\bar{\nu}_e} E_e \cos\theta} \tag{10}$$

   and show that

$$\mathrm{d}\Gamma_\mu \; = \; \left(\int_{|E_{\bar{\nu}_e} - E_e|}^{|E_{\bar{\nu}_e} + E_e|} \delta(m_\mu - E_{\bar{\nu}_e} - u - E_e) \mathrm{d}u \right) \frac{\overline{|\mathcal{M}|^2} \mathrm{d}E_{\bar{\nu}_e}}{16(2\pi)^4 m_\mu} \frac{\mathrm{d}\vec{p'}}{E_e} \tag{11}$$

   Finally, by analyzing when the δ function in equation (11) is not zero, show that

$$\mathrm{d}\Gamma_\mu \; = \; \left(\int_{\frac{1}{2}m_\mu - E_e}^{\frac{1}{2}m_\mu} \frac{\overline{|\mathcal{M}|^2}}{16(2\pi)^4 m_\mu} \mathrm{d}E_{\bar{\nu}_e} \right) \frac{\mathrm{d}\vec{p'}}{E_e} \tag{12}$$

6. Using equations (6) and (12), show that the decay rate per unit of energy of the emitted electron is

$$\frac{\mathrm{d}\Gamma_\mu}{\mathrm{d}E_e} \; = \; \frac{32\, G_F^2\, m_\mu}{(4\pi)^3} E_e^2 \left(\frac{m_\mu}{2} - \frac{2}{3} E_e \right) \tag{13}$$

   Hint: convert the integral over $\vec{p'}$ into an integral over E_e. Plot this function and give the most likely energy of the emitted electron.

7. Compute the total lifetime of the muon. Experimental measurements give $m_\mu = 105.65$ MeV and $\tau_{\text{life}} = 2.20\, 10^{-6}$ s.

- Deduce the value of the Fermi constant.
- In a historical perspective, let's assume that an underlying dynamic is responsible of this decay. Write G_F as a function of the mass of the new force mediator m_X and its coupling constant g_X.
- Assuming the same strength that the electromagnetic interaction ($g_X = g_{\mathrm{EM}}$), compute the mass of the new force mediator.
- Given the mass of the W boson, deduce the value of $\alpha_{\mathrm{weak}} = g_W^2/4\pi$. Compare with α_{EM} and comment on the "weakness" of the weak interaction.