12. Wichtige Verteilungen

12.1 Binomialverteilung

12.2 Poisson-Verteilung

12.3 Gauß-Verteilung (Normalverteilung)

12.4 Zentraler Grenzwertsatz

12.5 Gleichverteilung

12.6 Breit-Wigner-Verteilung

12.7 Faltung von Verteilungen

Binomialverteilung

Beispiel ①: Münze (Kopf oder Zahl)

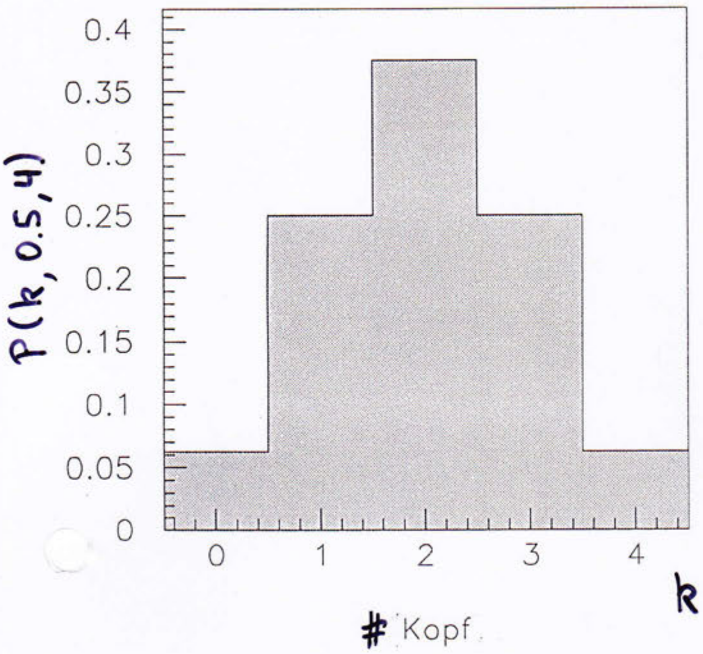
$p = 0.5$

Wie groß ist die W'keit in vier Würfen 3 mal Kopf zu erhalten?

$n = 4$

$k = 3$

$$P(3;\, 0.5, 4) = \binom{4}{3} \cdot (0.5)^3 \cdot (1-0.5)^1 = \frac{4!}{3!\,1!}\,(0.5)^4 = \underline{0.25}$$

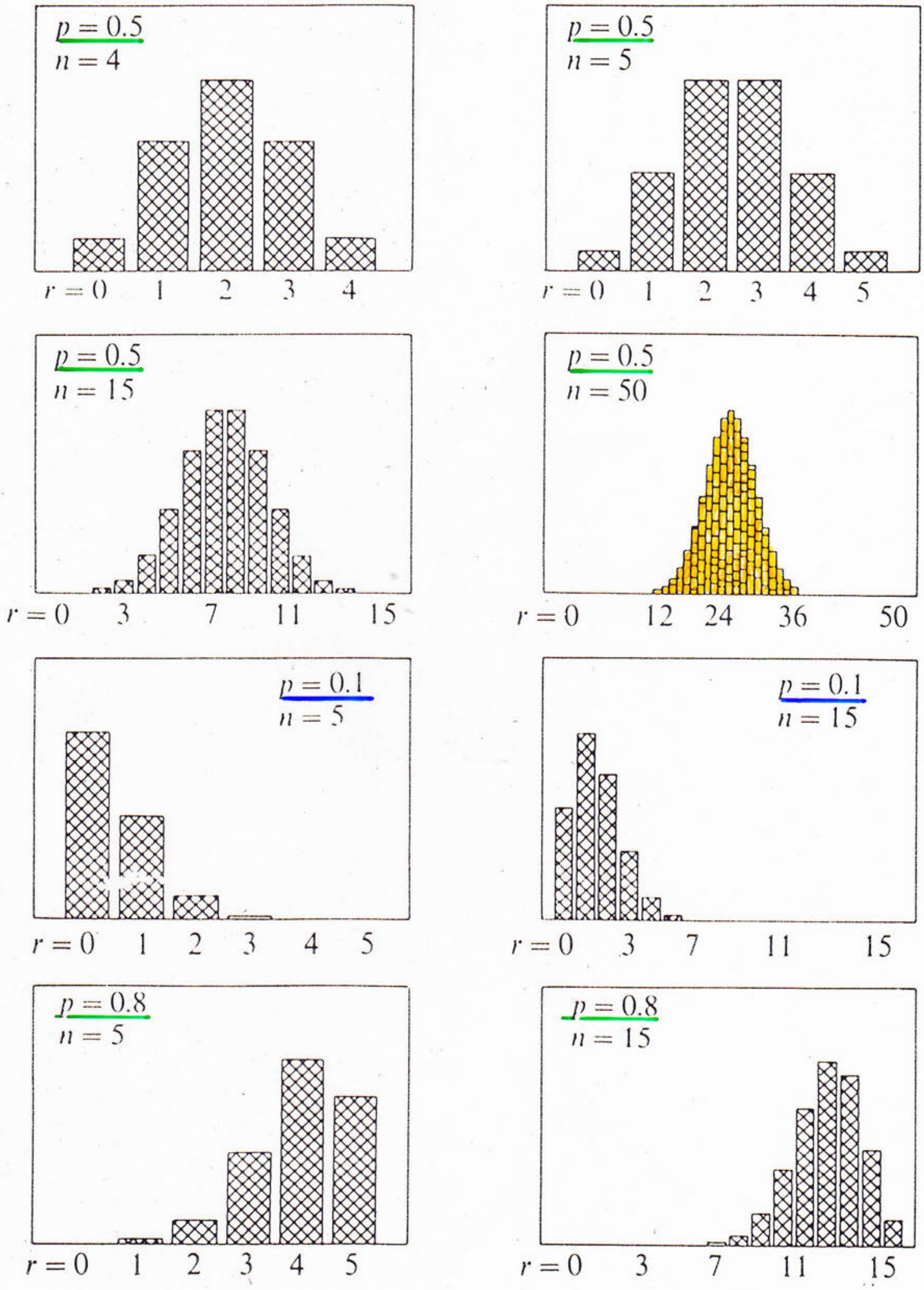


k=4: $P(K,K,K,K) = (0.5)^4 = \underline{0.0625}$

k=3: $P(K,K,K,Z) + P(K,K,Z,K) + P(K,Z,K,K) + P(Z,K,K,K)$

$$= 4 \cdot (0.5)^4 = \underline{0.25}$$

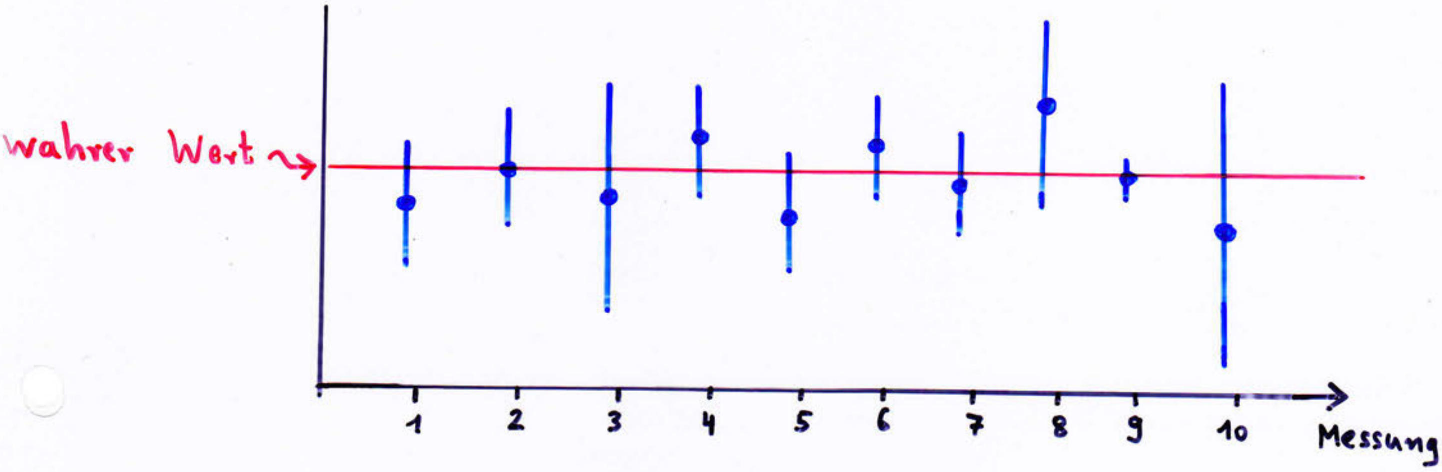
Beispiele für Binomialverteilungen für verschiedene Werte von n und p.



Binomialverteilung

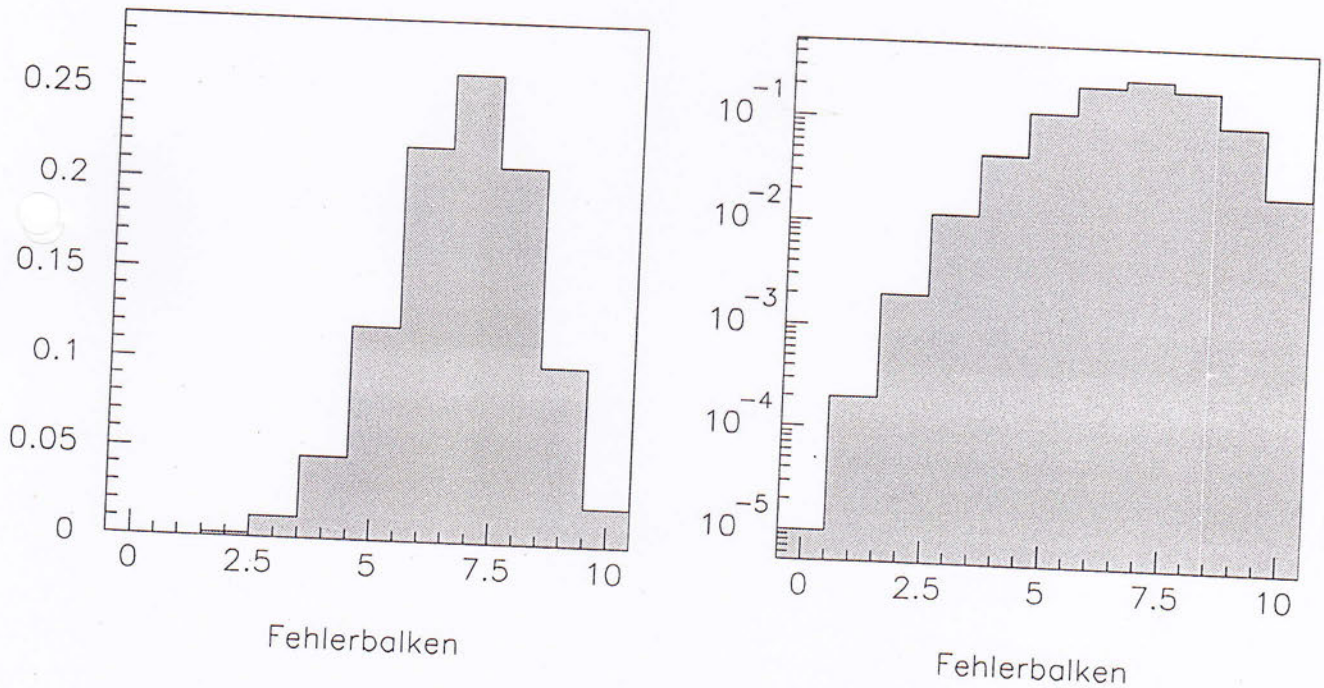
Beispiel ③: Experimentelle Meßung, Fehlerbalken;

10 Experimente messen eine phys. Größe



i. allg. liegt der 'wahre Wert' mit einer W'keit von $P = 0.683$ im Fehlerintervall.

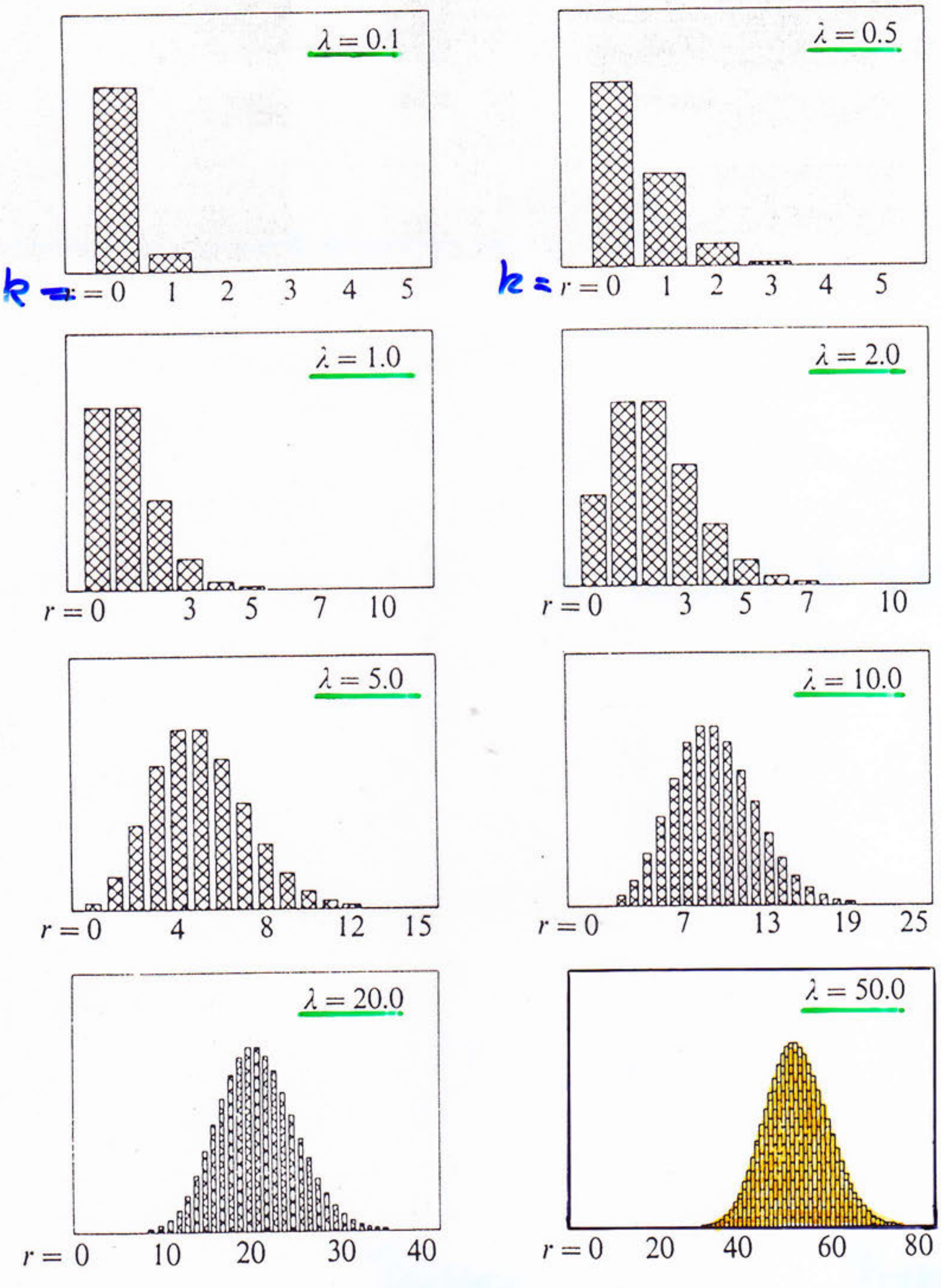
Wie groß ist die W'keit, daß der wahre Wert innerhalb der Fehlerbalken aller 10 Expt. liegt?



$$P(\underset{}{10};\ \underset{p}{0.683},\ \underset{n}{10}) = (0.683)^{10} = \underline{\underline{0.022}}$$

$\Rightarrow$ Fehler überschätzt ?!

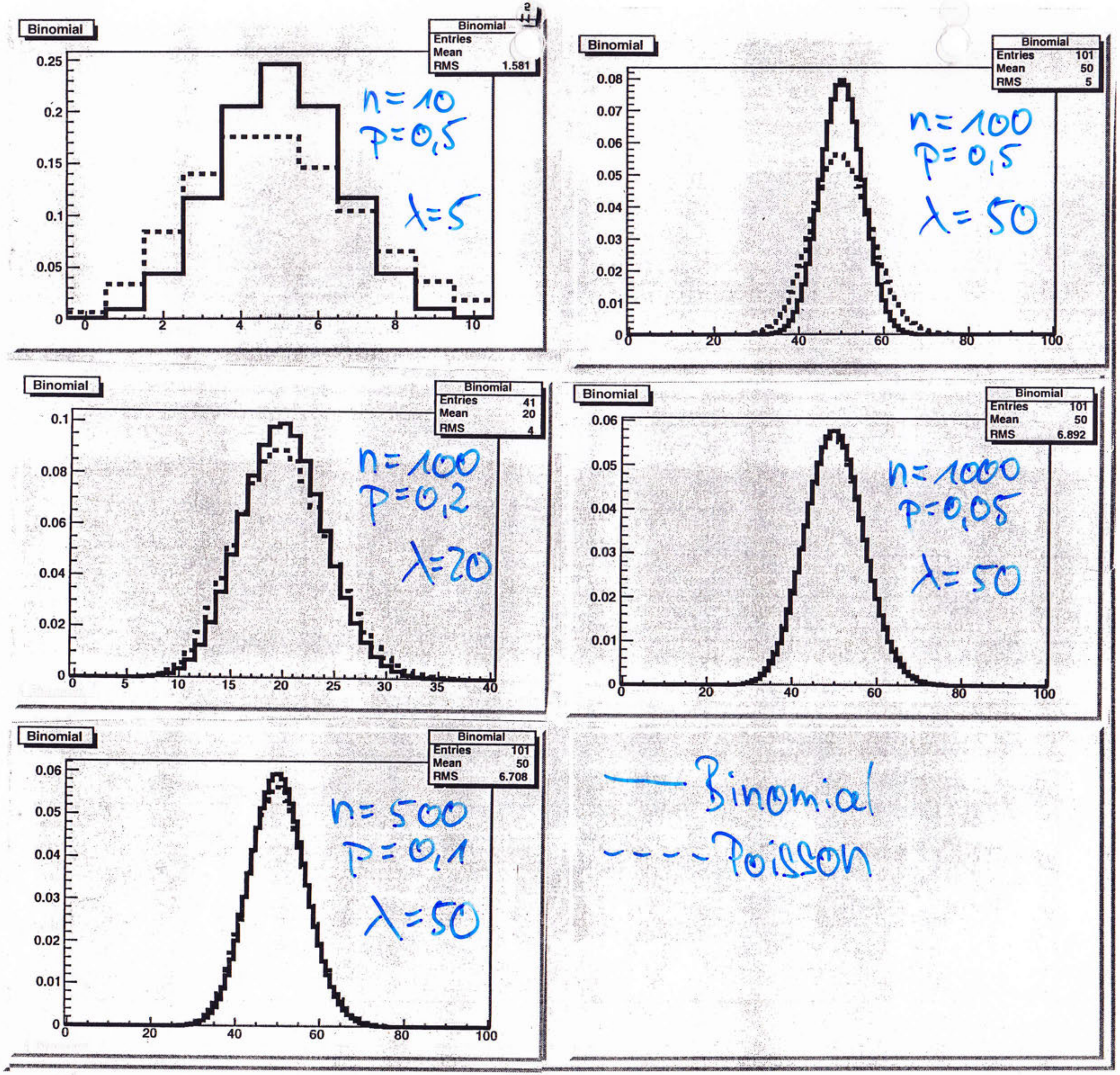
Poisson verteilung



$\lambda < 1$ → wahrscheinlichster Wert $k = 0$

$1 < \lambda \lesssim 5$ → wahrscheinlichster Wert $k = \lambda$, $k = \lambda - 1$
$P(k) = P(k-1)$ — λ ganzzahlig

$\lambda \gtrsim 20$ → symmetrische Verteilung
↳ Grenzwert - Verteilung


Binomial
Binomial
Entries
Mean
RMS 1.581
n=10
p=0,5
λ=5
Binomial
Binomial
Entries 101
Mean 50
RMS 5
n=100
p=0,5
λ=50
Binomial
Binomial
Entries 41
Mean 20
RMS 4
n=100
p=0,2
λ=20
Binomial
Binomial
Entries 101
Mean 50
RMS 6.892
n=1000
p=0,05
λ=50
Binomial
Binomial
Entries 101
Mean 50
RMS 6.708
n=500
p=0,1
λ=50
— Binomial
---- Poisson

Poisson - Verteilung

Beispiel ① Radioaktiver Zerfall

n := Zahl der Versuche = Zahl der vorhandenen Atomkerne

λ = Mittlere Zahl der im Zeitintervall Δt zerfallenden Kerne

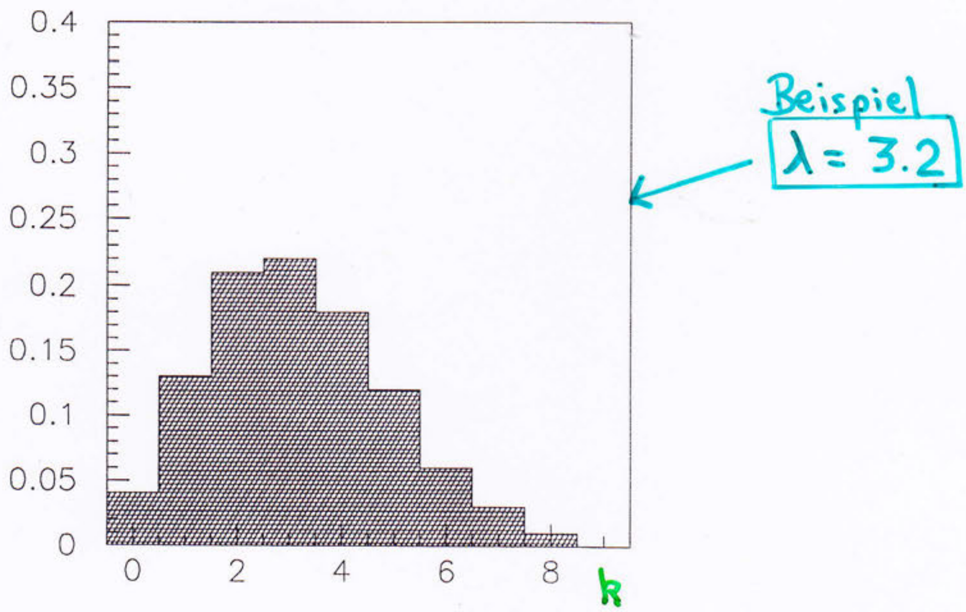
gesucht: W'keit dafür, daß k Kerne im Zeitintervall Δt zerfallen

$\lambda = n \cdot p(\Delta t)$

$p(\Delta t)$ = W'keit für Zerfall eines einzelnen Kerns im Intervall Δt.

Erwartung: Poisson-Verteilung mit $\boxed{\lambda = n \cdot p(\Delta t)}$

$$P(k,\lambda) = \frac{\lambda^k \cdot e^{-\lambda}}{k!} \qquad k = 0, 1, 2, \ldots$$



Poisson prob, rad. decay, lam=3.2

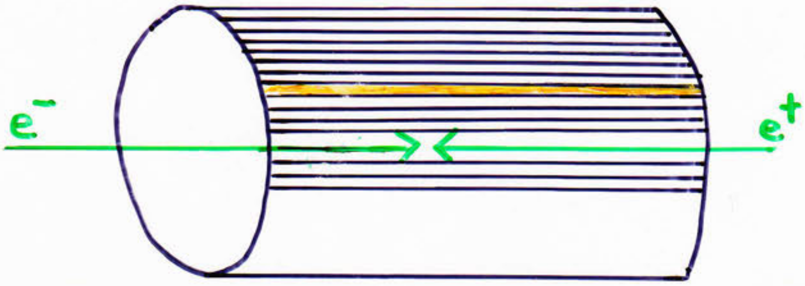
$f(0) = 0.04$
$f(1) = 0.13$
$f(2) = 0.21$
$f(3) = 0.22$
$f(4) = 0.18$
$f(5) = 0.12$
$f(6) = 0.06$
$f(7) = 0.03$
$f(8) = 0.01$
$f(9) = 0.00$

$\overline{1.00}$

↳ Evidenz für stat. Natur des radioaktiven Zerfalls

2. Beispiel für die Approximation einer Binomialverteilung durch eine Poisson-Verteilung

Experiment der Teilchenphysik



50 Szintillationszähler (Detektoren)

Meßzeit 1 Jahr $p(\text{Ausfall}) = 0.01$ (pro Zähler)

Wie groß ist die W'keit, daß am Ende der Meßperiode noch alle Zähler funktionieren?

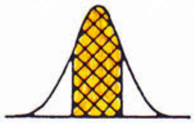
Berechne: Zahl der ausgefallenen Zähler

$$P(0; 0.01, 50) = \binom{50}{0} \cdot p^0 (1-p)^{50} = (0.99)^{50} = \underline{0.605}$$

Poisson: $k=0 \Rightarrow P(0) = e^{-\lambda} = e^{-50\cdot 0.01} = \underline{\underline{0.607}}$

	N = 50		N = 100	
	Binomial	Poisson	Binomial	Poisson
0 ausgef.	0.605	0.607	0.366	0.368
1 ausgef.	0.306	0.303	0.370	0.368
2 ausgef.	0.076	0.076	0.185	0.184
≥3 ausgef.	0.013	0.014	0.079	0.080

TABLE 3.2
TWO-TAILED GAUSSIAN INTEGRAL
Giving the percentage probability that a point lies within the given number of σ from the mean



$\frac{x-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	0.00	0.80	1.60	2.39	3.19	3.99	4.78	5.58	6.38	7.17
0.10	7.97	8.76	9.55	10.34	11.13	11.92	12.71	13.50	14.28	15.07
0.20	15.85	16.63	17.41	18.19	18.97	19.74	20.51	21.28	22.05	22.82
0.30	23.58	24.34	25.10	25.86	26.61	27.37	28.12	28.86	29.61	30.35
0.40	31.08	31.82	32.55	33.28	34.01	34.73	35.45	36.16	36.88	37.59
0.50	38.29	38.99	39.69	40.39	41.08	41.77	42.45	43.13	43.81	44.48
0.60	45.15	45.81	46.47	47.13	47.78	48.43	49.07	49.71	50.35	50.98
0.70	51.61	52.23	52.85	53.46	54.07	54.67	55.27	55.87	56.46	57.05
0.80	57.63	58.21	58.78	59.35	59.91	60.47	61.02	61.57	62.11	62.65
0.90	63.19	63.72	64.24	64.76	65.28	65.79	66.29	66.80	67.29	67.78
1.00	68.27	68.75	69.23	69.70	70.17	70.63	71.09	71.54	71.99	72.43
1.10	72.87	73.30	73.73	74.15	74.57	74.99	75.40	75.80	76.20	76.60
1.20	76.99	77.37	77.75	78.13	78.50	78.87	79.23	79.59	79.95	80.29
1.30	80.64	80.98	81.32	81.65	81.98	82.30	82.62	82.93	83.24	83.55
1.40	83.85	84.15	84.44	84.73	85.01	85.29	85.57	85.84	86.11	86.38
1.50	86.64	86.90	87.15	87.40	87.64	87.89	88.12	88.36	88.59	88.82
1.60	89.04	89.26	89.48	89.69	89.90	90.11	90.31	90.51	90.70	90.90
1.70	91.09	91.27	91.46	91.64	91.81	91.99	92.16	92.33	92.49	92.65
1.80	92.81	92.97	93.12	93.28	93.42	93.57	97.71	93.85	93.99	94.12
1.90	94.26	94.39	94.51	94.64	94.76	94.88	95.00	95.12	95.23	95.34
2.00	95.45	95.56	95.66	95.76	95.86	95.96	96.06	96.15	96.25	96.34
2.10	96.43	96.51	96.60	96.68	96.76	96.84	96.92	97.00	97.07	97.15
2.20	97.22	97.29	97.36	97.43	97.49	97.56	97.62	97.68	97.74	97.80
2.30	97.86	97.91	97.97	98.02	98.07	98.12	98.17	98.22	98.27	98.32
2.40	98.36	98.40	98.45	98.49	98.53	98.57	98.61	98.65	98.69	98.72
2.50	98.76	98.79	98.83	98.86	98.89	98.92	98.95	98.98	99.01	99.04
2.60	99.07	99.09	99.12	99.15	99.17	99.20	99.22	99.24	99.26	99.29
2.70	99.31	99.33	99.35	99.37	99.39	99.40	99.42	99.44	99.46	99.47
2.80	99.49	99.50	99.52	99.53	99.55	99.56	99.58	99.59	99.60	99.61
2.90	99.63	99.64	99.65	99.66	99.67	99.68	99.69	99.70	99.71	99.72
3.00	99.73	99.74	99.75	99.76	99.76	99.77	99.78	99.79	99.79	99.80
3.10	99.81	99.81	99.82	99.83	99.83	99.84	99.84	99.85	99.85	99.86
3.20	99.86	99.87	99.87	99.88	99.88	99.88	99.89	99.89	99.90	99.90
3.30	99.90	99.91	99.91	99.91	99.92	99.92	99.92	99.92	99.93	99.93
3.40	99.93	99.94	99.94	99.94	99.94	99.94	99.94	99.95	99.95	99.95
3.50	99.95	99.96	99.96	99.96	99.96	99.96	99.96	99.96	99.97	99.97

TABLE 3.3
ONE-TAILED GAUSSIAN INTEGRAL
Giving the percentage probability that a point lies within the given number of σ to one side of the mean

$\frac{x-\mu}{\sigma}$	0.00	0.01	0.02	0.03	0.04	0.05	0.06	0.07	0.08	0.09
0.00	50.00	50.40	50.80	51.20	51.60	51.99	52.39	52.79	53.19	53.59
0.10	53.98	54.38	54.78	55.17	55.57	55.96	56.36	56.75	57.14	57.53
0.20	57.93	58.32	58.71	59.10	59.48	59.87	60.26	60.64	61.03	61.41
0.30	61.79	62.17	62.55	62.93	63.31	63.68	64.06	64.43	64.80	65.17
0.40	65.54	65.91	66.28	66.64	67.00	67.36	67.72	68.08	68.44	68.79
0.50	69.15	69.50	69.85	70.19	70.54	70.88	71.23	71.57	71.90	72.24
0.60	72.57	72.91	73.24	73.57	73.89	74.22	74.54	74.86	75.17	75.49
0.70	75.80	76.11	76.42	76.73	77.04	77.34	77.64	77.94	78.23	78.52
0.80	78.81	79.10	79.39	79.67	79.95	80.23	80.51	80.78	81.06	81.33
0.90	81.59	81.86	82.12	82.38	82.64	82.89	83.15	83.40	83.65	83.89
1.00	84.13	84.38	84.61	84.85	85.08	85.31	85.54	85.77	85.99	86.21
1.10	86.43	86.65	86.86	87.08	87.29	87.49	87.70	87.90	88.10	88.30
1.20	88.49	88.69	88.88	89.07	89.25	89.44	89.62	89.80	89.97	90.15
1.30	90.32	90.49	90.66	90.82	90.99	91.15	91.31	91.47	91.62	91.77
1.40	91.92	92.07	92.22	92.36	92.51	92.65	92.79	92.92	93.06	93.19
1.50	93.32	93.45	93.57	93.70	93.82	93.94	94.06	94.18	94.29	94.41
1.60	94.52	94.63	94.74	94.84	94.95	95.05	95.15	95.25	95.35	95.45
1.70	95.54	95.64	95.73	95.82	95.91	95.99	96.08	96.16	96.25	96.33
1.80	96.41	96.49	96.56	96.64	96.71	96.78	96.86	96.93	96.99	97.06
1.90	97.13	97.19	97.26	97.32	97.38	97.44	97.50	97.56	97.61	97.67
2.00	97.72	97.78	97.83	97.88	97.93	97.98	98.03	98.08	98.12	98.17
2.10	98.21	98.26	98.30	98.34	98.38	98.42	98.46	98.50	98.54	98.57
2.20	98.61	98.64	98.68	98.71	98.75	98.78	98.81	98.84	98.87	98.90
2.30	98.93	98.96	98.98	99.01	99.04	99.06	99.09	99.11	99.13	99.16
2.40	99.18	99.20	99.22	99.25	99.27	99.29	99.31	99.32	99.34	99.36
2.50	99.38	99.40	99.41	99.43	99.45	99.46	99.48	99.49	99.51	99.52
2.60	99.53	99.55	99.56	99.57	99.59	99.60	99.61	99.62	99.63	99.64
2.70	99.65	99.66	99.67	99.68	99.69	99.70	99.71	99.72	99.73	99.74
2.80	99.74	99.75	99.76	99.77	99.77	99.78	99.79	99.79	99.80	99.81
2.90	99.81	99.82	99.82	99.83	99.84	99.84	99.85	99.85	99.86	99.86
3.00	99.87	99.87	99.87	99.88	99.88	99.89	99.89	99.89	99.90	99.90
3.10	99.90	99.91	99.91	99.91	99.92	99.92	99.92	99.92	99.93	99.93
3.20	99.93	99.93	99.94	99.94	99.94	99.94	99.94	99.95	99.95	99.95
3.30	99.95	99.95	99.95	99.96	99.96	99.96	99.96	99.96	99.96	99.97
3.40	99.97	99.97	99.97	99.97	99.97	99.97	99.97	99.97	99.97	99.98
3.50	99.98	99.98	99.98	99.98	99.98	99.98	99.98	99.98	99.98	99.98

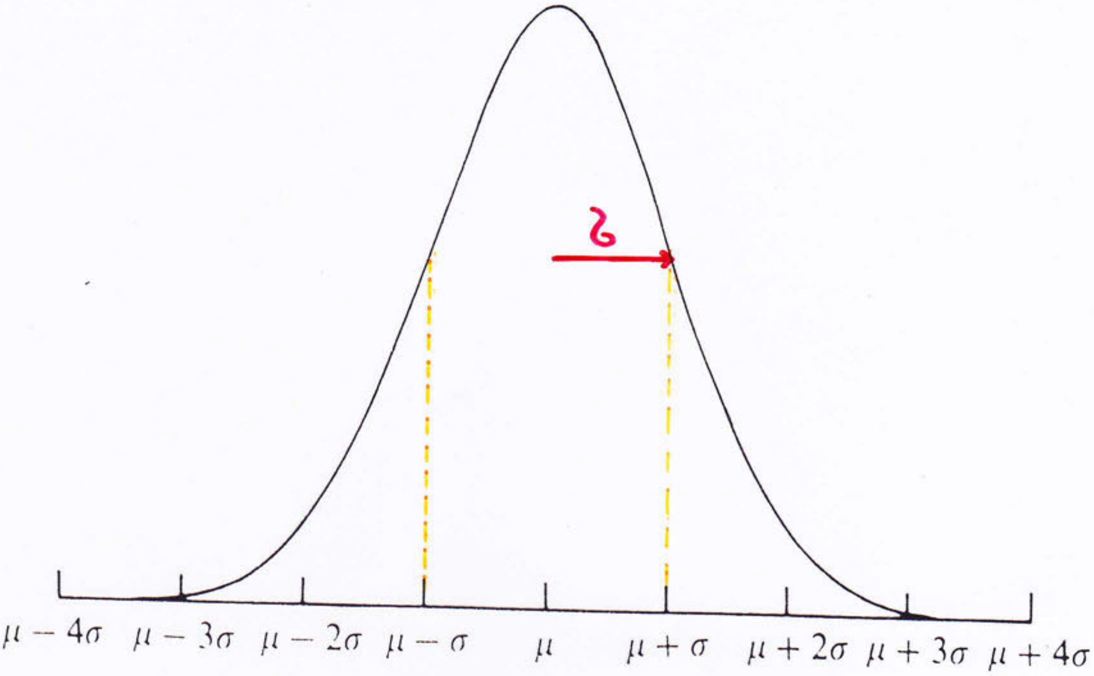


Fig. 3.3. The Gaussian distribution.

TABLE 3.1
USEFUL INTEGRALS

$$\boxed{\int_{-\infty}^{\infty} e^{-ax^2}\,dx \quad \sqrt{\frac{\pi}{a}}} \qquad \int_{-\infty}^{\infty} e^{-z^2/2}\,dz = \sqrt{2\pi}$$

$$\boxed{\int_{0}^{\infty} x\,e^{-ax^2}\,dx = \frac{1}{2a}} \qquad \int_{0}^{\infty} z\,e^{-z^2/2}\,dz = 1$$

$$\int_{-\infty}^{\infty} x^2 e^{-ax^2}\,dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \qquad \int_{-\infty}^{\infty} z^2\,e^{-z^2/2}\,dz = \sqrt{2\pi}.$$

Higher powers can be obtained by differentiating these with respect to a, giving

$$\int_{0}^{\infty} x^{2n+1}\,e^{-ax^2}\,dx = \frac{n!}{2a^{n+1}} \qquad \int_{0}^{\infty} z^{2n+1}\,e^{-z^2/2}\,dz = 2^n n!$$

$$\int_{-\infty}^{\infty} x^{2n}\,e^{-ax^2}\,dx = \frac{1.3.5\ldots(2n-1)}{2^n a^n}\sqrt{\frac{\pi}{a}}$$

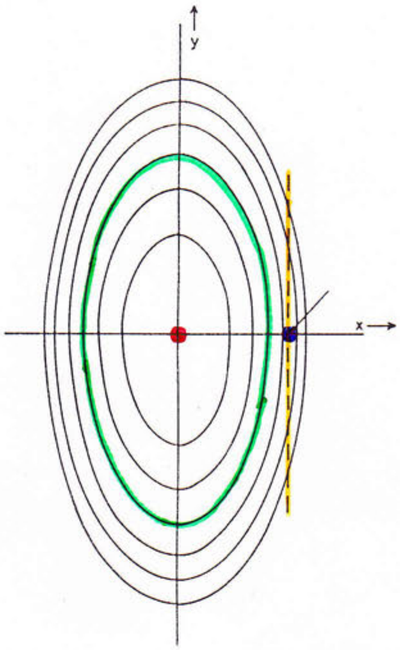
$$\int_{-\infty}^{\infty} z^{2n}\,e^{-z^2/2}\,dz = 1.3.5\ldots(2n-1)\sqrt{2\pi}.$$

For any odd power, the symmetric integral vanishes:

$$\int_{-\infty}^{\infty} x^{2n+1}\,e^{-ax^2}\,dx = \int_{-\infty}^{\infty} z^{2n+1}\,e^{-z^2/2}\,dz = 0.$$

Konturlinien für die Gaußverteilung in zwei Dimensionen

– Kovarianzellipsen –



unabhängige Variablen x, y

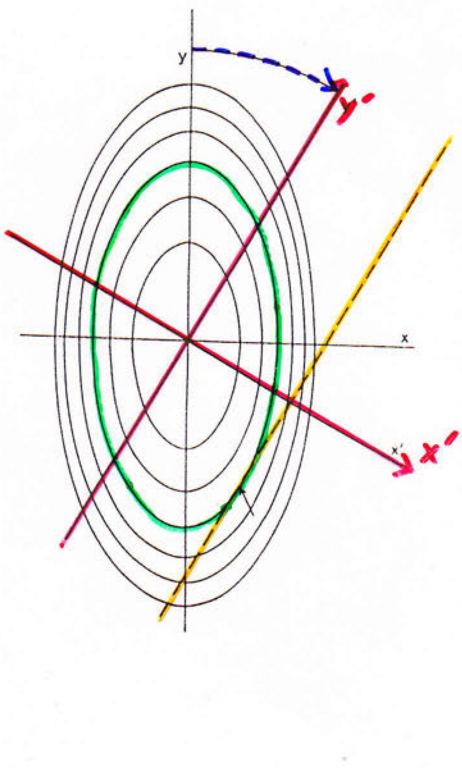
- $P_{max} = P(0,0)$

— $\boxed{\dfrac{x^2}{\sigma_x^2} + \dfrac{y^2}{\sigma_y^2} = 1}$

W'keit $= P(0,0) / \sqrt{e} = 0.607 \cdot P(0,0)$

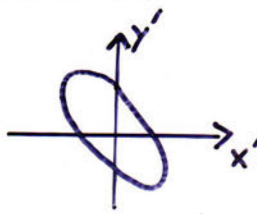
- Für alle x' gilt:

  $\max P(x', y) = P(x', y)\big|_{y=0}$



Übergang zu abhängigen Variablen x', y' (z.B. durch Rotation)

↳ Korrelationen



- Für $\tilde{x} > 0$ gilt:

  $\max P(\tilde{x}, y') = P(\tilde{x}, \tilde{y})$ $\boxed{\tilde{y} < 0}$

Für $\tilde{x} < 0$ gilt

  $\max P(\tilde{x}, y') = P(\tilde{x}, \tilde{y})$ $\boxed{\tilde{y} > 0}$

↳ neg. Korrelation

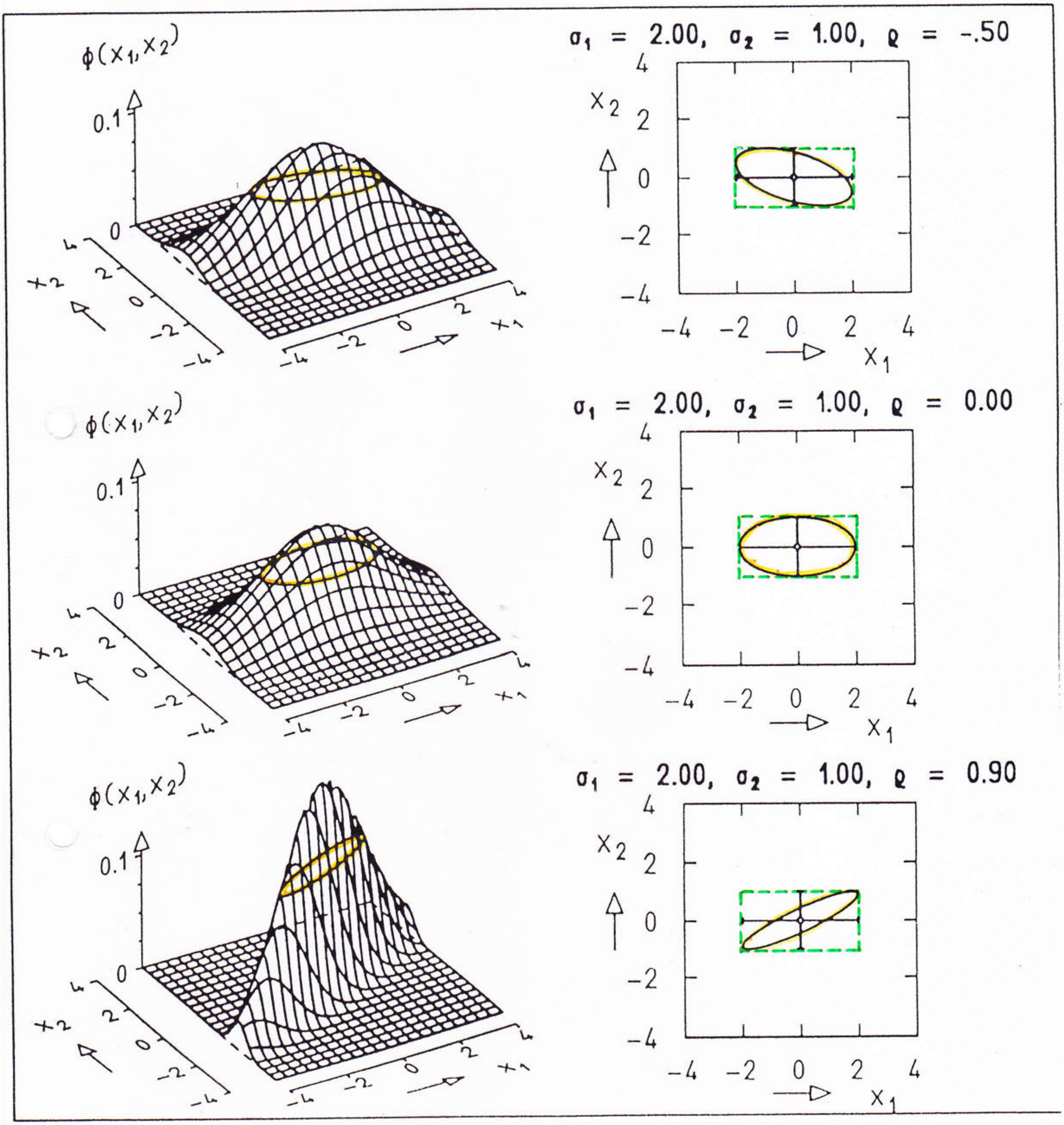
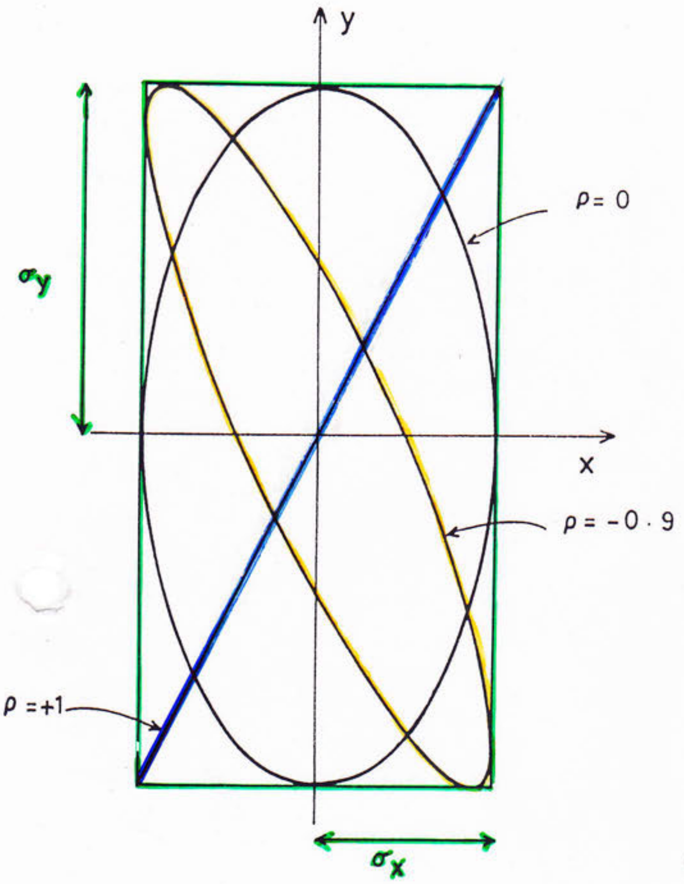


Bild 5.12: Wahrscheinlichkeitsdichte einer Gauß-Verteilung zweier Variabler (links) und zugehörige Kovarianzellipse (rechts). Die drei Zeilen des Bildes unterscheiden sich nur durch den Zahlwert des Korrelationskoeffizienten ρ.

Horizontalschnitte durch W'keitsdichte → konzentrische Ellipsen
Vertikalschnitte durch (0,0) → Gauß-Verteilungen

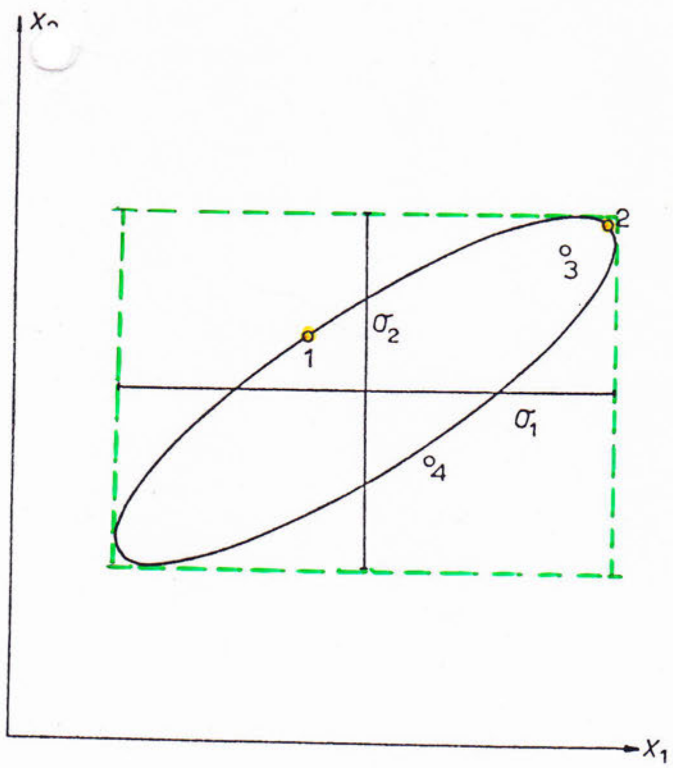


$\sigma_x, \sigma_y = \text{const}$

Variation von ρ

Neigungswinkel ϕ:

$$\boxed{\tan 2\phi = \frac{2 \cdot \rho\, \sigma_x \cdot \sigma_y}{\sigma_x^2 - \sigma_y^2}}$$



Wahrscheinlichkeits-Relationen

$$P(1) = P(2) = \frac{P_{max}}{\sqrt{e}}$$

$$P(3) > P(1)$$

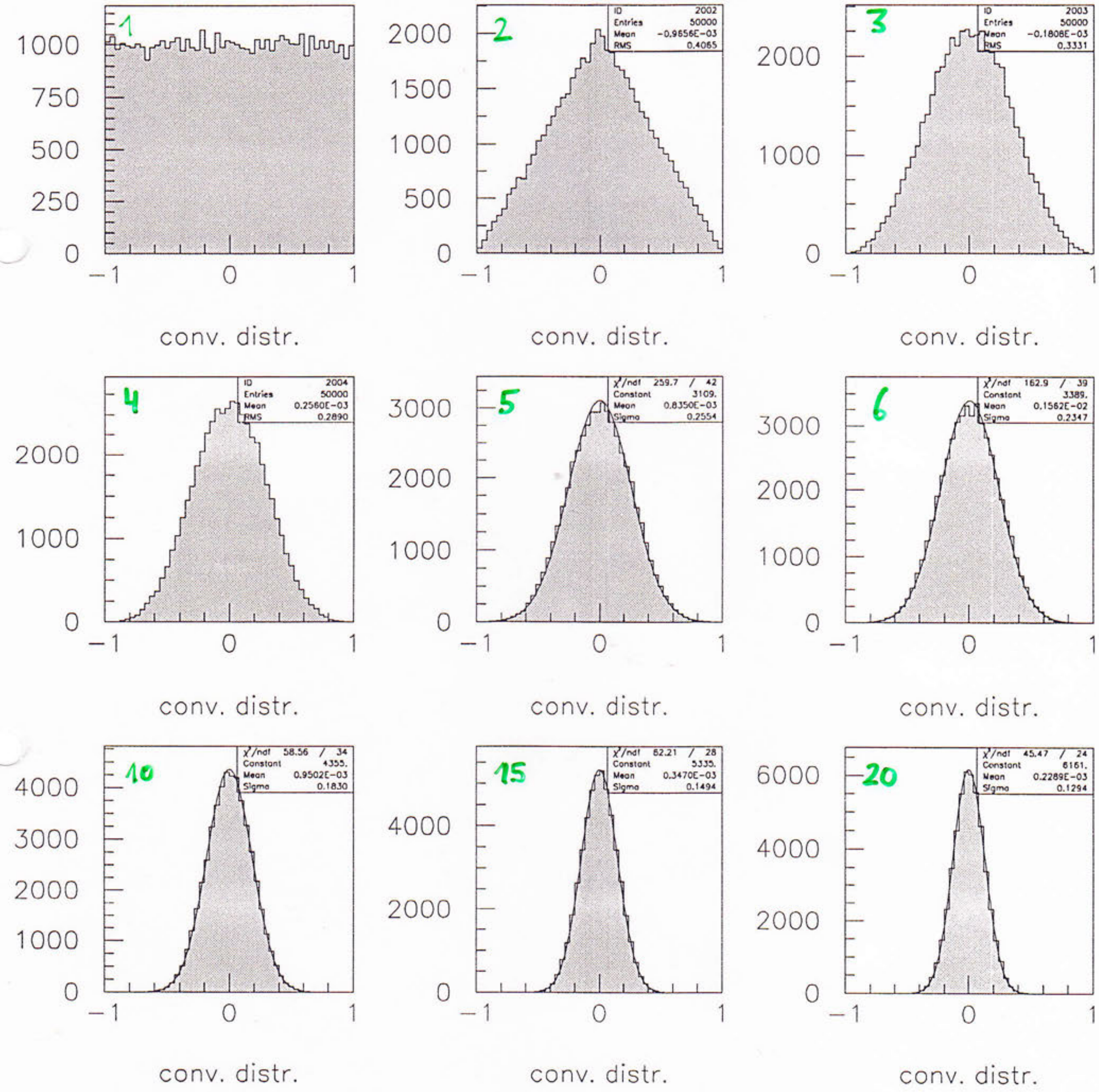
$$P(4) < P(1)$$

Linien gleicher W'keit
= Ellipsen, konzentrisch um Kovarianzellipse
innerhalb: größere W'keit
außerhalb: kleinere W'keit

W'keit einen Wert (x,y) innerhalb der Kovarianzellipse zu beobachten:

$$\int_A P(x,y)\,dx\,dy = 1 - e^{-1/2} = \underline{\underline{0.393}}$$

Beispiel: Faltung von Gleichverteilungen



n = Zahl der Zufallszahlen

$$\sigma_{theo} = \frac{b-a}{\sqrt{12}} = 0.577$$

	$\sigma_{theo}/\sqrt{n}$	Gauß-σ
5	0.258	0.255
10	0.183	0.183
15	0.149	0.149
20	0.129	0.129

Verteilung der Augenzahl

für n-Würfel

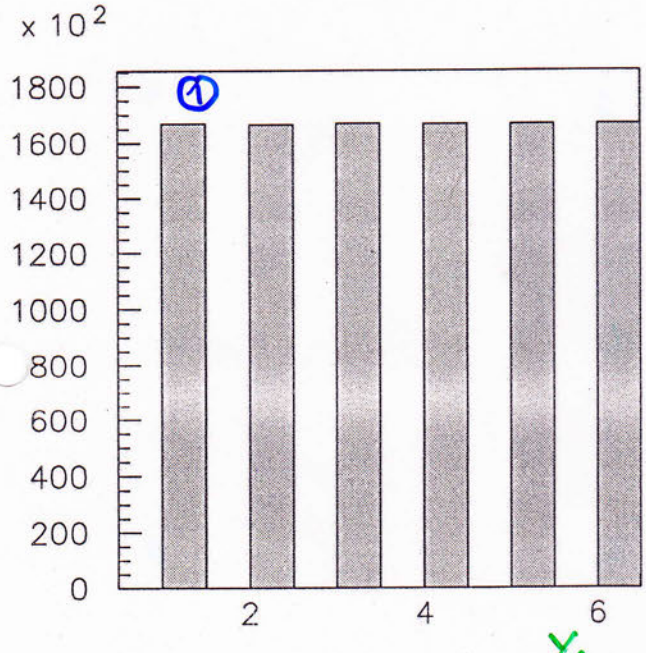
$x_i \in \{1,2,3,4,5,6\}$

$Y_1 = x_1$

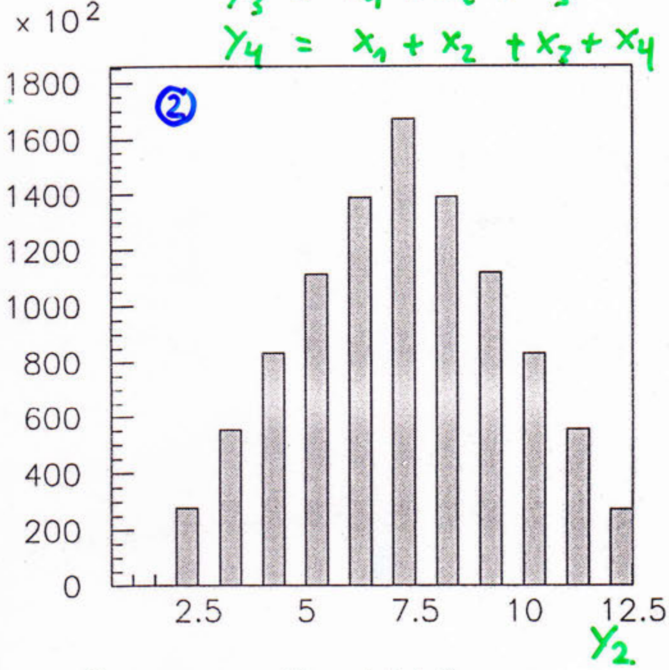
$Y_2 = x_1 + x_2$

$Y_3 = x_1 + x_2 + x_3$

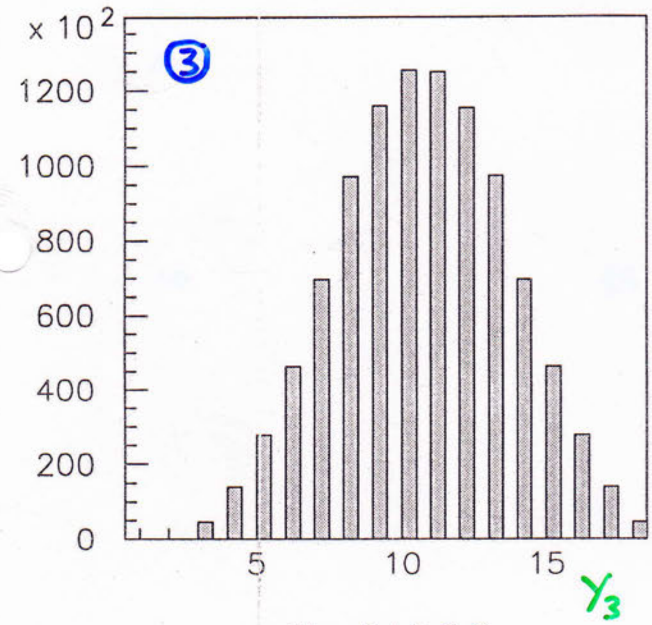
$Y_4 = x_1 + x_2 + x_3 + x_4$



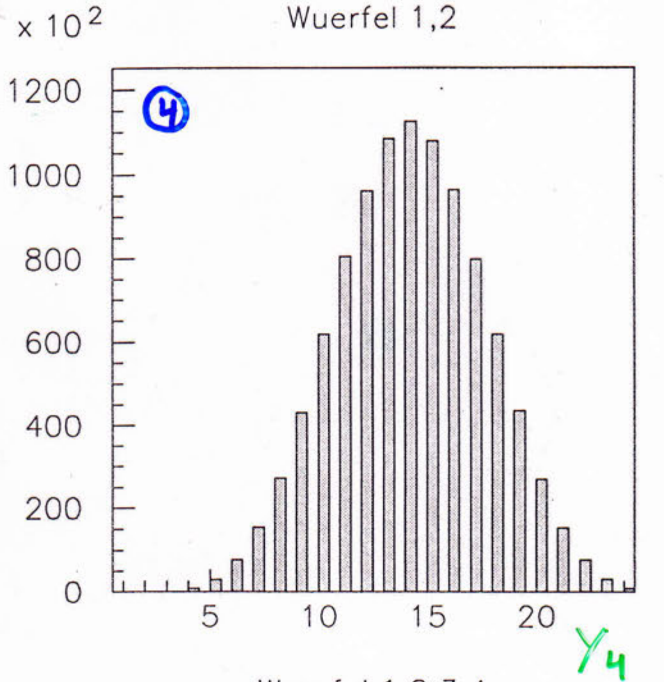
Wuerfel 1



Wuerfel 1,2

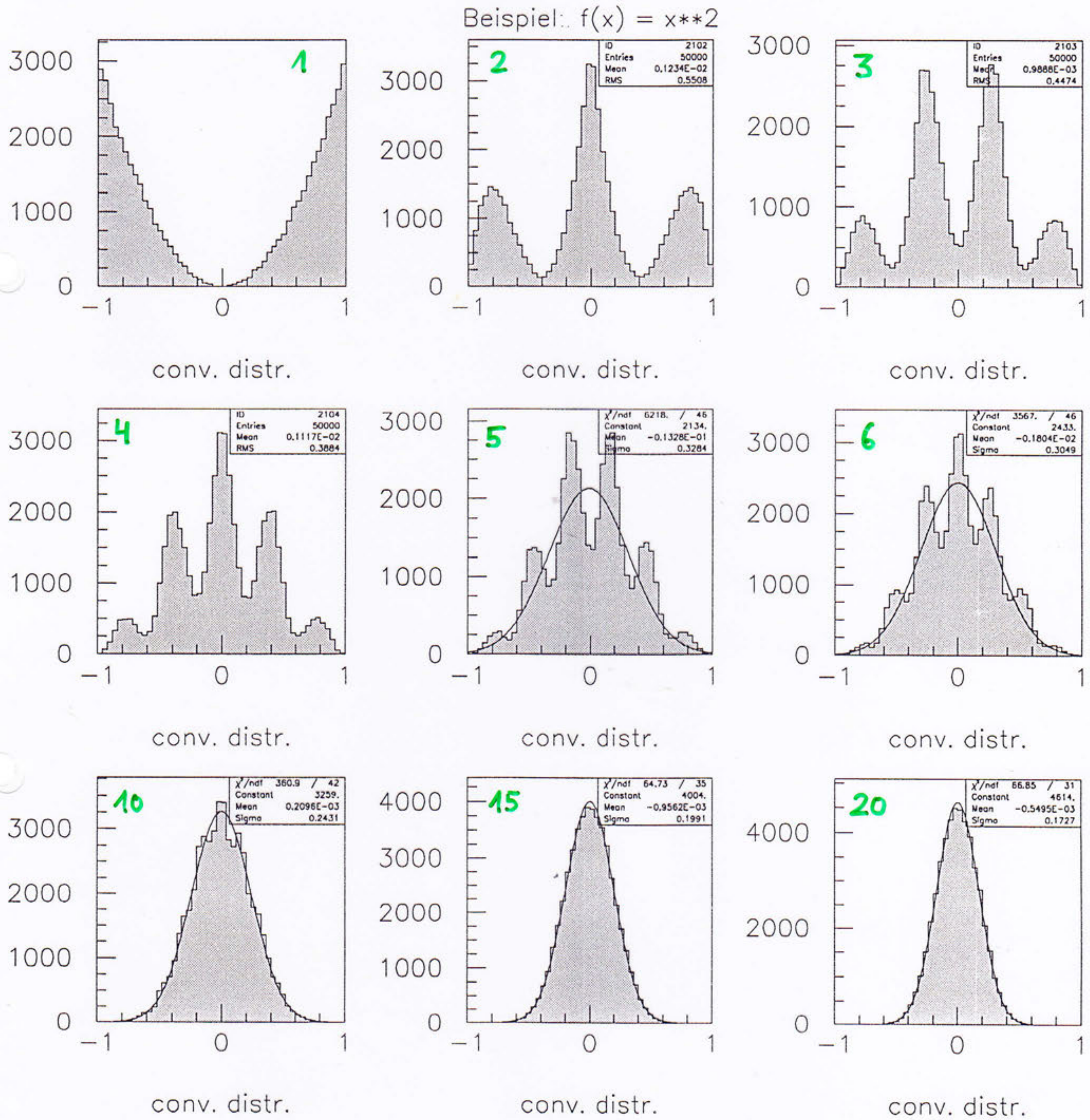


Wuerfel 1,2,3

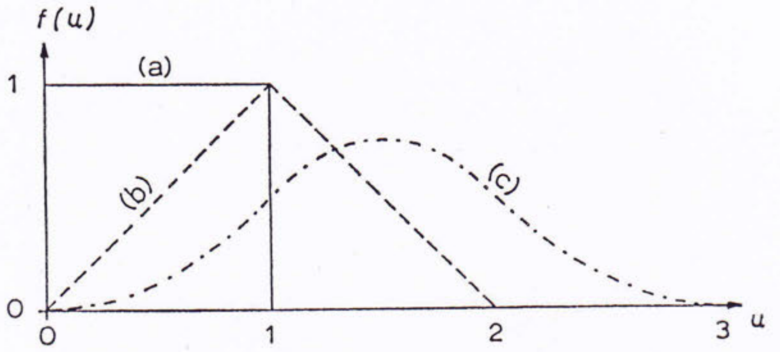


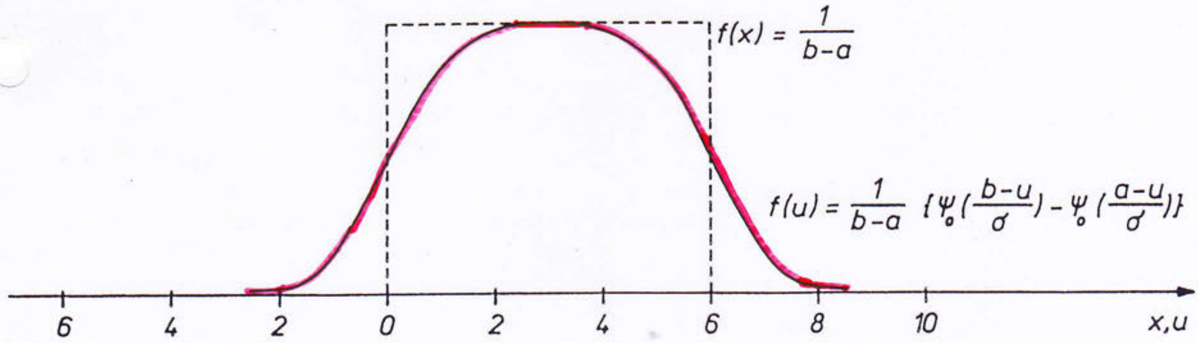
Wuerfel 1,2,3,4

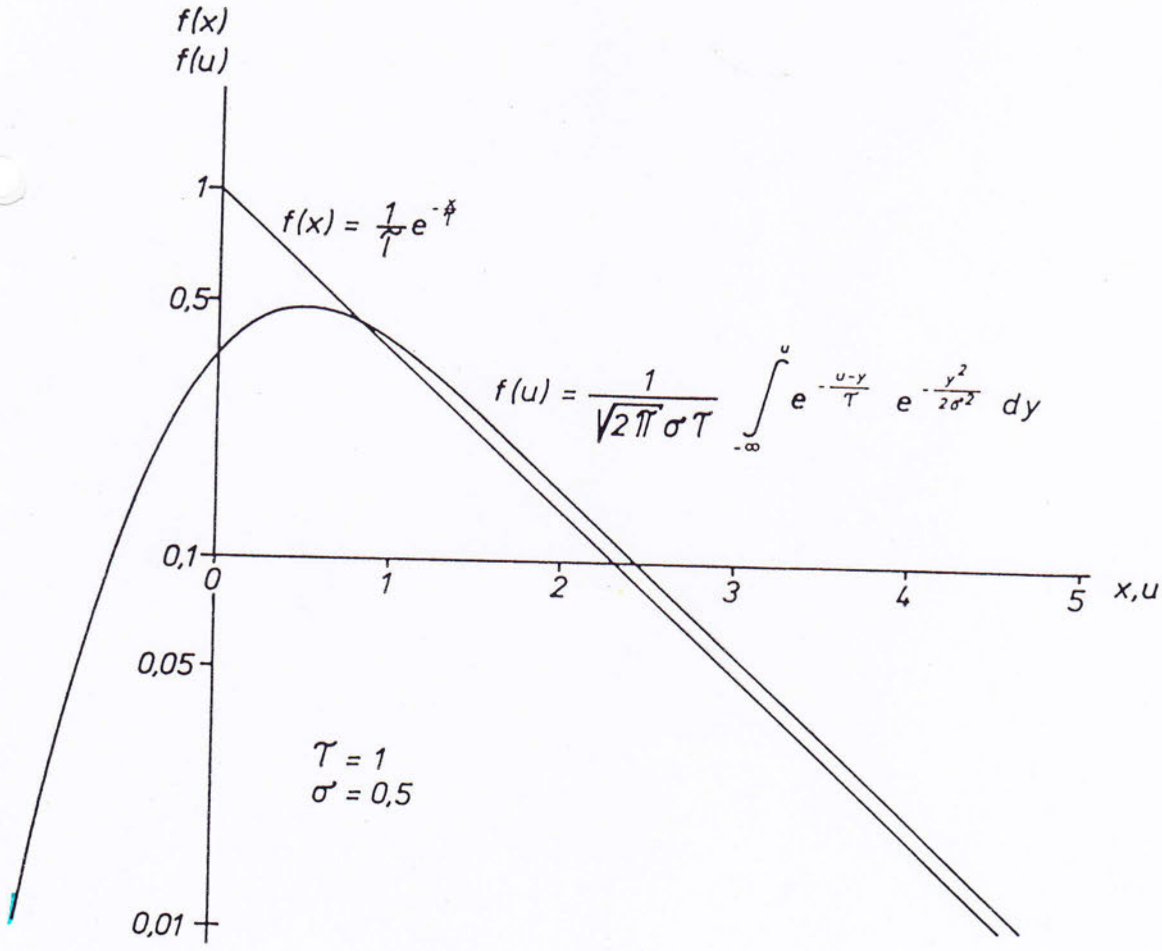
n Zufallszahlen aus einer $f(x) = x^2$ - Verteilung



n = Zahl der Zufallszahlen


f(u)
1
(a)
(b)
(c)
0
0
1
2
3
u


$f(x) = \frac{1}{b-a}$
$f(u) = \frac{1}{b-a} \{\psi_0(\frac{b-u}{\sigma}) - \psi_0(\frac{a-u}{\sigma})\}$
6
4
2
0
2
4
6
8
10
x,u


f(x)
f(u)
1
0,5
0,1
0,05
0,01
$f(x) = \frac{1}{\tau} e^{-\frac{x}{\tau}}$
$f(u) = \frac{1}{\sqrt{2\pi}\,\sigma\tau} \int_{-\infty}^{u} e^{-\frac{u-y}{\tau}} \; e^{-\frac{y^2}{2\sigma^2}} \, dy$
0
1
2
3
4
5
x,u
$\tau = 1$
$\sigma = 0{,}5$