14. Schätzung von Parametern

14.1 Forderungen an eine Schätzung

14.2 Beispiele für Stichprobenfunktionen

14.3 Die χ^2-Verteilung

14.4 Die Methode der kleinsten Quadrate

14.5 Die *Maximum Likelihood* Methode

14.6 Konfidenzintervalle

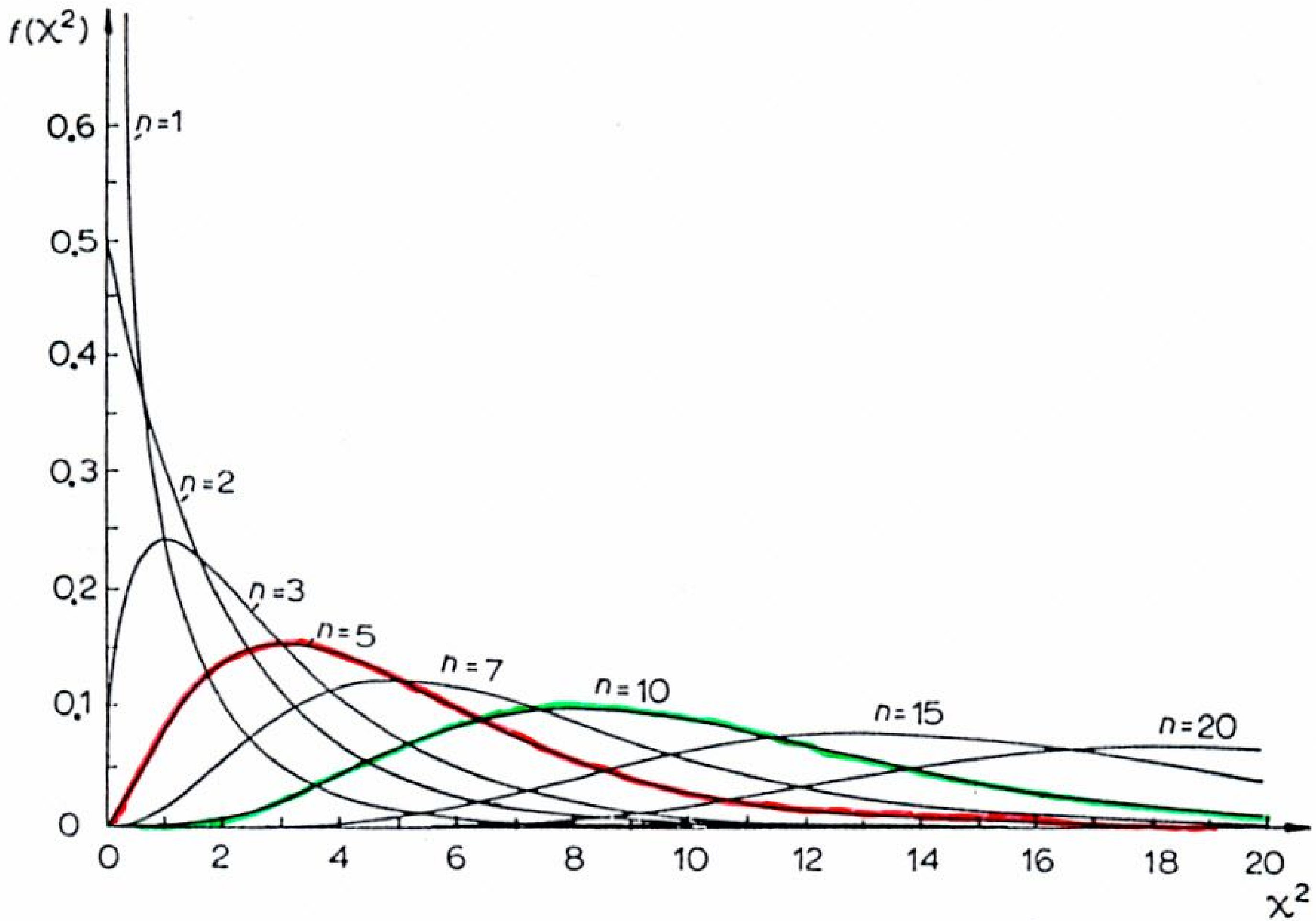


Bild 6.2 Wahrscheinlichkeitsdichte von χ^2.

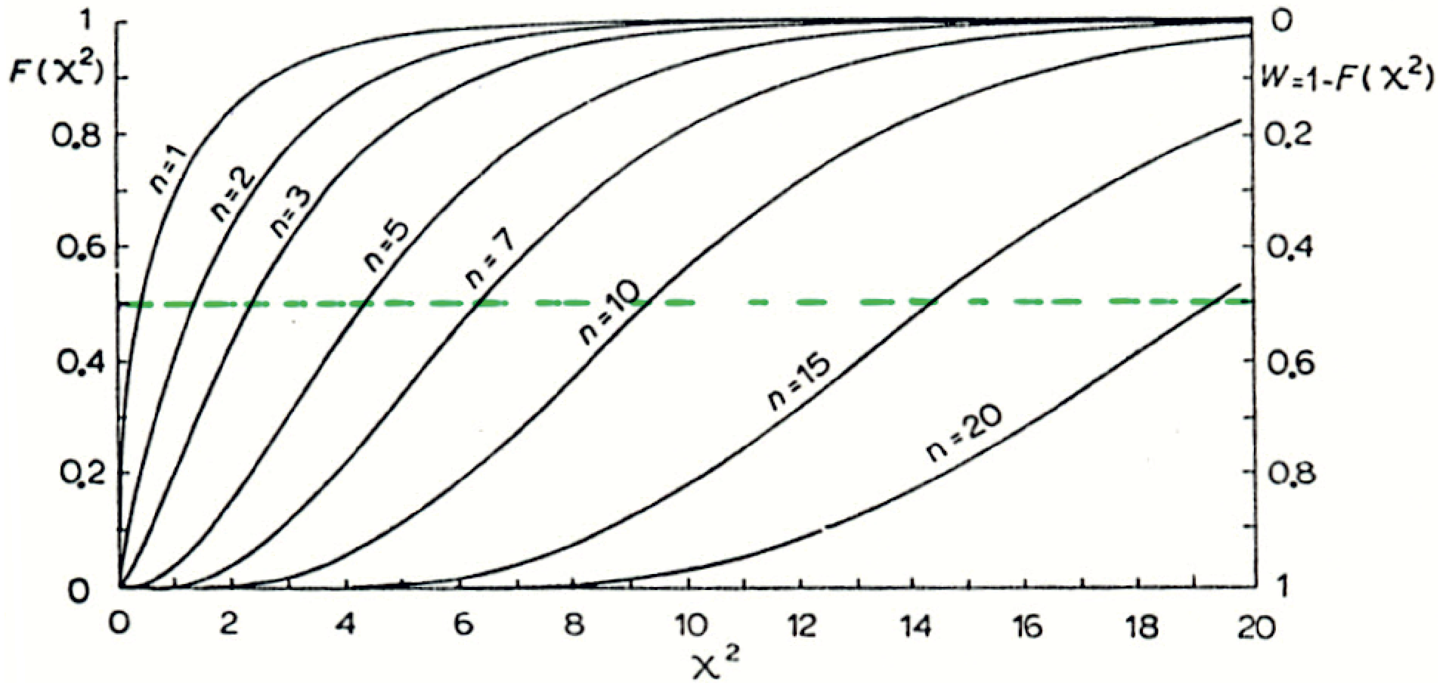
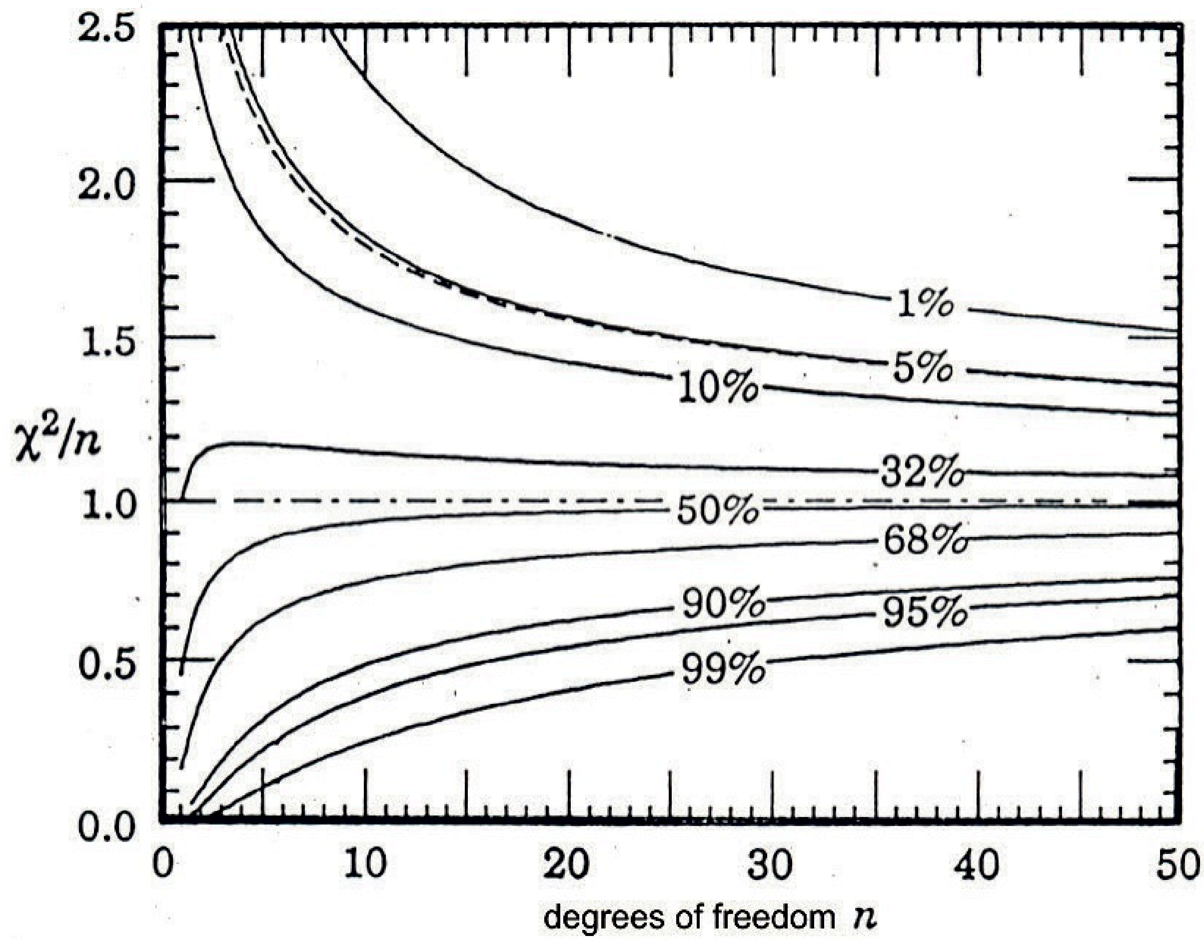


Bild 6.3 Verteilungsfunktion von χ^2.

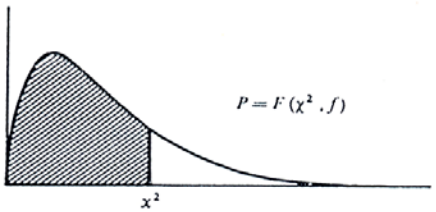


2.5
2.0
1.5
1.0
0.5
0.0
χ^2/n
0
10
20
30
40
50
degrees of freedom n
1%
5%
10%
32%
50%
68%
90%
95%
99%

Tafel F – 4
χ^2–Verteilung.

Tabelliert sind die Werte P, definiert durch

$$P = F(\chi^2, f) = \frac{1}{\Gamma(\frac{1}{2}f)\, 2^{\frac{1}{2}f}} \int_0^{\chi^2} u^{\frac{1}{2}f-1} e^{-\frac{1}{2}u}\, du$$

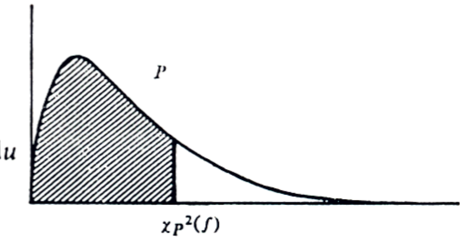


χ^2 \ f	1	2	3	4	5	6	7	8	9	10
0.1	0.248	0.049	0.008	0.001	0.000	0.000	0.000	0.000	0.000	0.000
0.2	0.345	0.095	0.022	0.005	0.001	0.000	0.000	0.000	0.000	0.000
0.3	0.416	0.139	0.040	0.010	0.002	0.001	0.000	0.000	0.000	0.000
0.4	0.473	0.181	0.060	0.018	0.005	0.001	0.000	0.000	0.000	0.000
0.5	0.520	0.221	0.081	0.026	0.008	0.002	0.001	0.000	0.000	0.000
0.6	0.561	0.259	0.104	0.037	0.012	0.004	0.001	0.000	0.000	0.000
0.7	0.597	0.295	0.127	0.049	0.017	0.006	0.002	0.000	0.000	0.000
0.8	0.629	0.330	0.151	0.062	0.023	0.008	0.003	0.001	0.000	0.000
0.9	0.657	0.362	0.175	0.075	0.030	0.011	0.004	0.001	0.000	0.000
1.0	0.683	0.393	0.199	0.090	0.037	0.014	0.005	0.002	0.001	0.000
1.1	0.706	0.423	0.223	0.106	0.046	0.01[illegible]	0.007	0.002	0.001	0.000
1.2	0.727	0.451	0.247	0.122	0.055	0.023	0.009	0.003	0.001	0.000
1.3	0.746	0.478	0.271	0.139	0.065	0.02[illegible]	0.012	0.004	0.002	0.001
1.4	0.763	0.503	0.294	0.156	0.076	0.034	0.014	0.006	0.002	0.001
1.5	0.779	0.528	0.318	0.173	0.087	0.041	0.018	0.007	0.003	0.00[illegible]
1.6	0.794	0.551	0.341	0.191	0.099	0.047	0.021	0.009	0.004	0.001
1.7	0.808	0.573	0.363	0.209	0.111	0.055	0.025	0.011	0.005	0.002
1.8	0.820	0.593	0.385	0.228	0.124	0.063	0.030	0.013	0.006	0.002
1.9	0.832	0.613	0.407	0.246	0.137	0.071	0.035	0.016	0.007	0.003
2.0	0.843	0.632	0.428	0.264	0.151	0.080	0.040	0.019	0.009	0.004
3.0	0.917	0.777	0.608	0.442	0.300	0.191	0.115	0.066	0.036	0.019
4.0	0.954	0.865	0.739	0.594	0.451	0.323	0.220	0.143	0.089	0.053
5.0	0.975	0.918	0.828	0.713	0.584	0.456	0.340	0.242	0.166	0.109
6.0	0.986	0.950	0.888	0.801	0.694	0.577	0.460	0.353	0.260	0.185
7.0	0.992	0.970	0.928	0.864	0.779	0.679	0.571	0.463	0.363	0.275
8.0	0.995	0.982	0.954	0.908	0.844	0.762	0.667	0.567	0.466	0.371
9.0	0.997	0.989	0.971	0.939	0.891	0.826	0.747	0.658	0.563	0.468
10.0	0.998	0.993	0.981	0.960	0.925	0.875	0.811	0.735	0.650	0.560
11.0	0.999	0.996	0.988	0.973	0.949	0.912	0.861	0.798	0.7[illegible]4	0.642
12.0	0.999	0.998	0.993	0.983	0.965	0.938	0.899	0.849	0.7[illegible]7	0.715
13.0	1.000	0.998	0.995	0.989	0.977	0.957	0.923	0.888	0.837	0.776
14.0	1.000	0.999	0.997	0.993	0.984	0.970	0.949	0.918	0.878	0.827
15.0	1.000	0.999	0.998	0.995	0.990	0.980	0.964	0.941	0.909	0.868
16.0	1.000	1.000	0.999	0.997	0.993	0.986	0.975	0.958	0.933	0.900
17.0	1.000	1.000	0.999	0.998	0.995	0.991	0.983	0.970	0.951	0.926
18.0	1.000	1.000	1.000	0.999	0.997	0.994	0.988	0.979	0.965	0.945
19.0	1.000	1.000	1.000	0.999	0.998	0.996	0.992	0.985	0.975	0.960
20.0	1.000	1.000	1.000	1.000	0.999	0.997	0.994	0.990	0.982	0.971

Tafel F – 5
Quantile der χ^2–Verteilung.

Tabelliert sind die Werte $\chi_P^2(f)$, definiert durch

$$P = \frac{1}{\Gamma(\frac{1}{2}f)\, 2^{\frac{1}{2}f}} \int_0^{\chi_P^2(f)} u^{-\frac{1}{2}f-1} e^{-\frac{1}{2}u}\, du$$



f \ P	0.100	0.500	0.700	0.800	0.900	0.950	0.990	0.995	0.999
1	0.02	0.45	1.07	1.64	2.71	3.84	6.63	7.88	10.83
2	0.21	1.39	2.41	3.22	4.61	5.99	9.21	10.60	13.82
3	0.58	2.37	3.66	4.64	6.25	7.81	11.34	12.84	16.27
4	1.06	3.36	4.88	5.99	7.78	9.49	13.28	14.86	18.47
5	1.61	4.35	6.06	7.29	9.24	11.07	15.09	16.75	20.51
6	2.20	5.35	7.23	8.56	10.64	12.59	16.81	18.55	22.46
7	2.83	6.35	8.38	9.80	12.02	14.07	18.48	20.28	24.32
8	3.49	7.34	9.52	11.03	13.36	15.51	20.09	21.95	26.12
9	4.17	8.34	10.66	12.24	14.68	16.92	21.67	23.59	27.88
10	4.87	9.34	11.78	13.44	15.99	18.31	23.21	25.19	29.59
11	5.58	10.34	12.90	14.63	17.27	19.68	24.72	26.76	31.26
12	6.30	11.34	14.01	15.81	18.55	21.03	26.22	28.30	32.91
13	7.04	12.34	15.12	16.98	19.81	22.36	27.69	29.82	34.53
14	7.79	13.34	16.22	18.15	21.06	23.68	29.14	31.32	36.12
15	8.55	14.34	17.32	19.31	22.31	25.00	30.58	32.80	37.70
16	9.31	15.34	18.42	20.47	23.54	26.30	32.00	34.27	39.25
17	10.09	16.34	19.51	21.61	24.77	27.59	33.41	35.72	40.79
18	10.86	17.34	20.60	22.76	25.99	28.87	34.81	37.16	42.31
19	11.65	18.34	21.69	23.90	27.20	30.14	36.19	38.58	43.82
20	12.44	19.34	22.77	25.04	28.41	31.41	37.57	40.00	45.31
21	13.24	20.34	23.86	26.17	29.61	32.67	38.93	41.40	46.80
22	14.04	21.34	24.94	27.30	30.81	33.92	40.29	42.80	48.27
23	14.85	22.34	26.02	28.43	32.01	35.17	41.64	44.18	49.73
24	15.66	23.34	27.10	29.55	33.20	36.42	42.98	45.56	51.18
25	16.47	24.34	28.17	30.68	34.38	37.65	44.31	46.93	52.62
26	17.29	25.34	29.25	31.79	35.56	38.89	45.64	48.29	54.05
27	18.11	26.34	30.32	32.91	36.74	40.11	46.96	49.64	55.48
28	18.94	27.34	31.39	34.03	37.92	41.34	48.28	50.99	56.89
29	19.77	28.34	32.46	35.14	39.09	42.56	49.59	52.34	58.30
30	20.60	29.34	33.53	36.25	40.26	43.77	50.89	53.67	59.70
40	29.05	39.33	44.17	47.27	51.80	55.76	63.70	66.76	73.39
50	37.69	49.33	54.72	58.16	63.17	67.51	76.16	79.49	86.66
60	46.46	59.33	65.23	68.97	74.40	79.08	88.38	91.95	99.61
70	55.33	69.33	75.69	79.71	85.53	90.53	100.43	104.21	112.32
80	64.28	79.33	86.12	90.40	96.58	101.88	112.33	116.32	124.84
90	73.29	89.33	96.52	101.05	107.56	113.15	124.12	128.30	137.21
100	82.36	99.33	106.91	111.67	118.50	124.34	135.81	140.17	149.45

Beispiel: Ausgleichsgerade, Matrixschreibweise

$$y(x) = m \cdot x + c$$
$$= a_1 + a_2 \cdot x \qquad \vec{\lambda} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix}$$

n Meßpunkte
2 Parameter

$$A = \begin{pmatrix} 1 & x_1 \\ 1 & x_2 \\ \vdots & \vdots \\ 1 & x_n \end{pmatrix}$$

Annahme:
gleiche Fehler σ für alle Meßpunkte, keine Korrelationen

$$\Rightarrow \quad C = \begin{pmatrix} \sigma^2 & 0 & \cdots & 0 \\ 0 & \sigma^2 & & \\ \vdots & & \ddots & \\ & & & \sigma^2 \end{pmatrix}$$

Lösung: $\vec{\lambda} = (A^t C^{-1} A)^{-1} A^t C^{-1} \cdot \vec{y}$

$$C = \sigma^2 \cdot \mathbb{I}d \Rightarrow \quad \vec{\lambda} = \sigma^2 (A^t A)^{-1} \frac{1}{\sigma^2} A^t \cdot \vec{y}$$

$$\Leftrightarrow \quad \vec{\lambda} = \begin{pmatrix} a_1 \\ a_2 \end{pmatrix} = \underbrace{\begin{pmatrix} \sum 1 & \sum x_i \\ \sum x_i & \sum x_i^2 \end{pmatrix}^{-1}} \begin{pmatrix} \sum y_i \\ \sum x_i \cdot y_i \end{pmatrix}$$

$$= \frac{1}{n(\overline{x^2} - \bar{x}^2)} \begin{pmatrix} \overline{x^2} & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \end{pmatrix}$$

$$\Rightarrow \quad \boxed{a_2 = \hat{m} = \frac{1}{n \cdot (\overline{x^2} - \bar{x}^2)} (-\bar{x} \cdot \sum y_i + 1 \cdot \sum x_i y_i) = \frac{-\bar{x}\bar{y} + \overline{xy}}{(\overline{x^2} - \bar{x}^2)}}$$

Fehlermatrix der Parameter $\vec{\lambda}$: $\quad C_\lambda = (A^t C^{-1} A)^{-1}$

$$= \sigma^2 (A^t A)^{-1} = \frac{\sigma^2}{n(\overline{x^2} - \bar{x}^2)} \begin{pmatrix} \overline{x^2} & -\bar{x} \\ -\bar{x} & 1 \end{pmatrix}$$

äquivalent zu Gleichungen aus Kap. 14.4.1

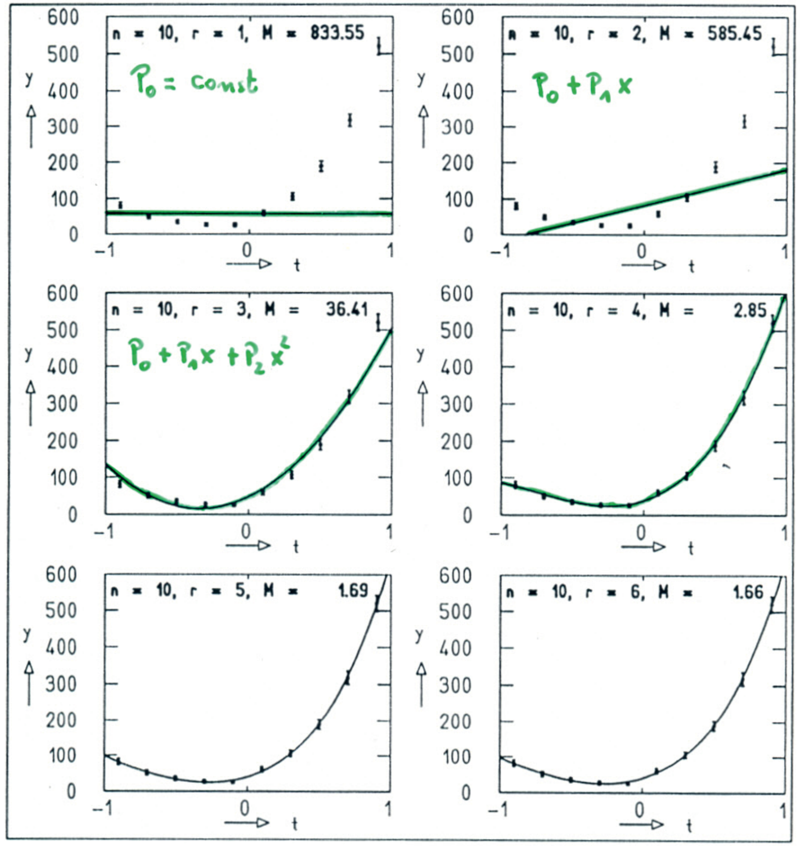
Beispiel: $\boxed{\text{Polynom-Anpassung}}$

$$y(x) = a_1 + a_2 \cdot x + a_3 \cdot x^2$$

$$A = \begin{pmatrix} 1 & x_1 & x_1^2 \\ 1 & x_2 & x_2^2 \\ \vdots & \vdots & \vdots \\ 1 & x_n & x_n^2 \end{pmatrix}$$

$$\vec{\lambda} = \begin{pmatrix} a_1 \\ a_2 \\ a_3 \end{pmatrix} = \begin{pmatrix} \sum 1 & \sum x_i & \sum x_i^2 \\ \sum x_i & \sum x_i^2 & \sum x_i^3 \\ \sum x_i^2 & \sum x_i^3 & \sum x_i^4 \end{pmatrix}^{-1} \begin{pmatrix} \sum y_i \\ \sum x_i y_i \\ \sum x_i^2 y_i \end{pmatrix}$$

Beispiel: Polynomanpassung an Messdaten



Fit-Parameter:

r	P_0	P_1	P_2	P_3	P_4	P_5	f	χ^2
1	57.85						9	833.55
2	82.66	99.10					8	585.45
3	47.27	185.96	273.61				7	36.41
4	37.94	126.55	312.02	137.59			6	2.85
5	39.62	119.10	276.49	151.91	52.60		5	1.68
6	39.88	121.39	273.19	136.58	56.90	16.72	4	1.66

\# Parameter

\# Freiheitsgrade

komplizierterer, nicht-linearer Fall:

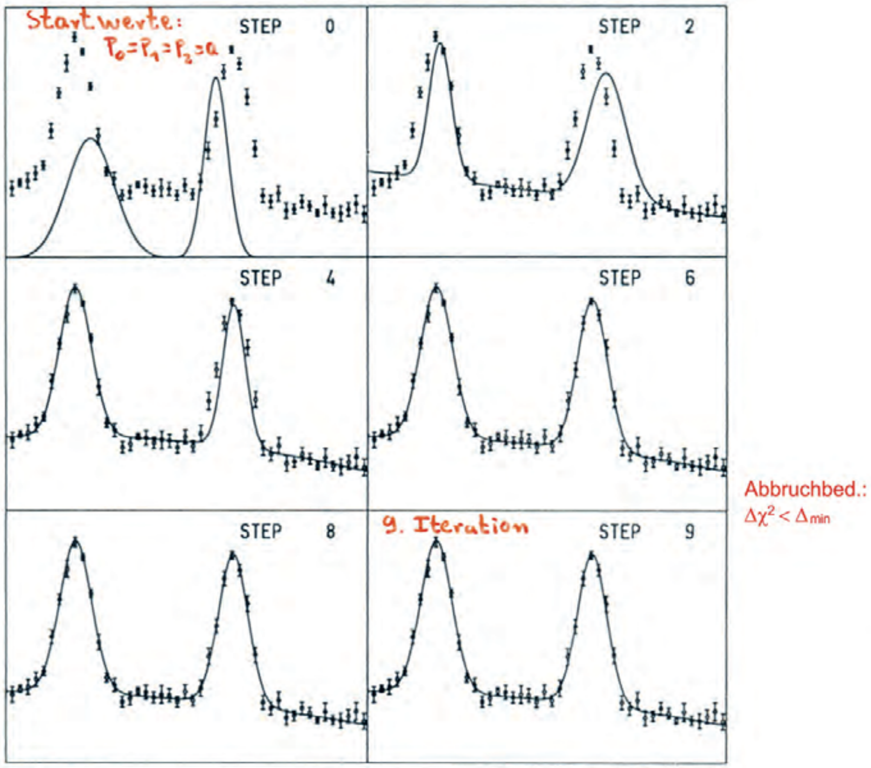


Bild 9.9: Schrittweise Annäherung der anzupassenden Funktion an die Meßwerte.

Zwei Signale : Gauß-Form

Untergrund : unbekannt, stetig → Polynom-Ansatz

$$f(x,\vec{\lambda}) = P_0 + P_1 \cdot x + P_2 \cdot x^2 + P_3 \cdot \exp\left\{-(P_4 - x)^2 / 2P_5^2\right\} + P_6 \cdot \exp\left\{-(P_7 - x)^2 / 2 \cdot P_8^2\right\}$$

$$\vec{\lambda} = \begin{pmatrix} P_0 \\ P_1 \\ \vdots \\ P_8 \end{pmatrix}$$

9 Parameter

$\Longrightarrow$ benutze zuverlässige, erprobte Computer-Programme zur Minimum-Bestimmung.

z.B. $\boxed{\text{MINUIT}}$ → Übung

Zusammenfassung Max. Likelihood Methode

Meßwerte: $x_1, x_2, \ldots\ldots, x_n$

Wahrscheinlichkeitsdichte: $f(x, \vec{\lambda})$

Parameter: $\vec{\lambda} = (\lambda_1, \lambda_2, \ldots\ldots, \lambda_m)$

Likelihood - Funktion: $\mathcal{L} = \prod_{i=1}^{n} f(x_i, \vec{\lambda})$

$$\ln \mathcal{L} = \sum_{i=1}^{n} \ln f(x_i, \vec{\lambda})$$

Max. Likelihood Schätzer $\hat{\lambda}$:
(globales Maximum der Likelihood Funktion)

$$\boxed{\left.\frac{d \ln \mathcal{L}}{d \lambda_j}\right|_{\hat{\lambda}_j} = 0 \qquad j = 1, 2, \ldots, m}$$

Eigenschaften des Max. Likelihood Schätzers:

(i) konsistent: $\lim_{n \to \infty} \sigma^2(\hat{\lambda}) = 0.$

(ii) Invariant unter Parametertransformationen

$\alpha \longrightarrow \beta(\alpha)$

$\hat{\alpha} \longrightarrow \beta(\hat{\alpha})$

Max. Likelihood Schätzer kann verzerrt sein:

$$E\{\hat{\lambda}\} = \int \hat{\lambda} \cdot \mathcal{L} \cdot dX = \lambda_0 (1 + \varepsilon(n))$$

$$\varepsilon(n) \to 0 \qquad n \to \infty.$$

(iii) Für $n \to \infty$ ist die max. Likelihood Schätzung effizient, d.h. die Varianz nimmt den kleinst möglichen Wert an

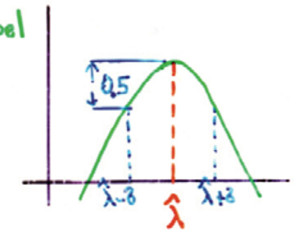
$$\mathrm{Var}(\hat{\lambda}) = - \frac{1}{E\left\{ \frac{d^2 \ln \mathcal{L}}{d\lambda^2} \Big|_{\hat{\lambda}} \right\}}$$

Cramér-Rao Bound.

II. Fehler der Parameter

Wenn $\frac{d^2 \ln \mathcal{L}}{d\lambda^2} = \mathrm{const}$ (z.B. Gaußverteilung)

$$\Rightarrow \ln \mathcal{L} = - \frac{(\lambda - \hat{\lambda})^2}{2b^2} + c$$ Parabel



1-σ Variation: $\Delta \ln \mathcal{L} = 1/2$

n-σ: $\Delta \ln \mathcal{L} = n^2/2$

auch dann, wenn $\ln \mathcal{L}$ nicht parabolisch wird diese Definition beibehalten

mehrere Parameter:

Kovarianzmatrix der Parameter aus den zweiten Ableitungen

$$a_{ij} = - \frac{\partial^2 \ln \mathcal{L}}{\partial \lambda_i \, \partial \lambda_j} \Big|_{\hat{\lambda}}$$

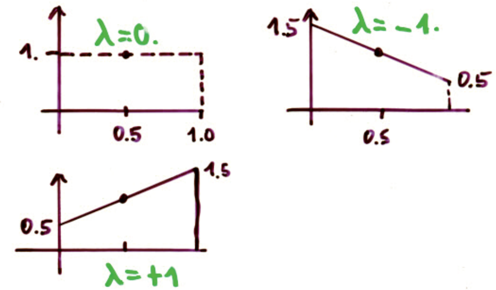
$$\boxed{C = A^{-1}}$$

7.2. Beispiele

Beispiel:

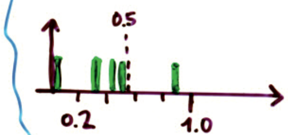
$$f(x;\lambda) = 1.0 + \lambda \cdot (x - 0.5)$$

$$\int_0^1 f(x,\lambda)\cdot dx = \underline{\underline{1.0}}$$



ges. Parameter: λ

beobachtete Werte: 0.89, 0.03, 0.50, 0.36, 0.49



$\lambda = +1.0 \quad \ln \mathcal{L} = \ln 1.39 + \ln 0.53 + \underline{\ln 1.0} + \ln 0.86 + \ln 0.99 = -0.47$

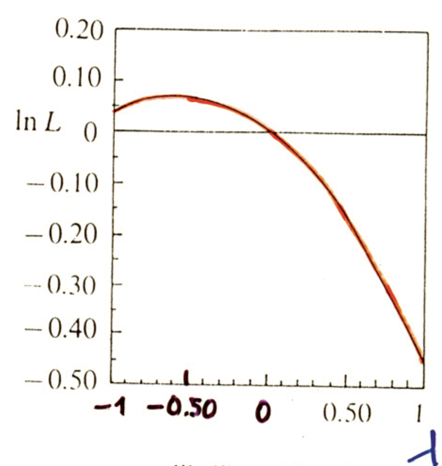
$\lambda = -0.5 \quad \ln \mathcal{L} = \ln 0.81 + \ln 1.24 + \underline{\ln 1.0} + \ln 1.07 + \ln 1.01 = 0.08$

$\lambda = -1.0 \quad \ln \mathcal{L} = \ln 0.61 + \ln 1.47 + \underline{\ln 1.0} + \ln 1.14 + \ln 1.01 = 0.03$

max. likelihood für $\lambda = -0.60$

$$\boxed{\hat{\lambda} = -0.60}$$

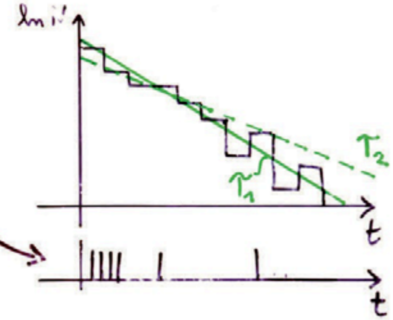
ML. – Schätzer



likelihood function.

Beispiel II: Lebensdauer eines Teilchens:

$$\boxed{f(t,\tau) = \frac{1}{\tau} \cdot e^{-t/\tau}}$$



n Meßwerte: $t_1, t_2, \ldots, t_n$

gesuchter Parameter:

Lebensdauer τ.

$$\ln \mathcal{L} = \sum_{i=1}^{n} \ln\left(\frac{1}{\tau} \cdot e^{-t_i/\tau}\right) = \sum_i \left(-\frac{t_i}{\tau} - \ln \tau\right)$$

$$\frac{d \ln \mathcal{L}}{d\tau} = \sum_{i=1}^{n} \left(\frac{t_i}{\tau^2} - \frac{1}{\tau}\right)$$

Maximum: $\left.\frac{d \ln \mathcal{L}}{d\tau}\right|_{\tau=\hat{\tau}} = 0 \iff \sum_{i=1}^{n} \left(\frac{t_i}{\hat{\tau}^2} - \frac{1}{\hat{\tau}}\right) = 0$

$$\Rightarrow \sum t_i/\hat{\tau}^2 = \frac{n}{\hat{\tau}}$$

$$\Rightarrow \boxed{\hat{\tau} = \frac{1}{n} \sum_{i=1}^{n} t_i}$$

Bemerkungen:

- Exp. Effekte, wie z.B. Auflösungsfunktion, können elegant mitberücksichtigt werden $\leftrightarrow$ Faltung

  $$f(t,\tau) \longrightarrow f'(t,\tau,\delta) = f(t,\tau) \otimes \underbrace{G(\delta)}_{\text{Gauß-Auflösung}}$$

- jedes Ereignis wird individuell berücksichtigt, kein Binning

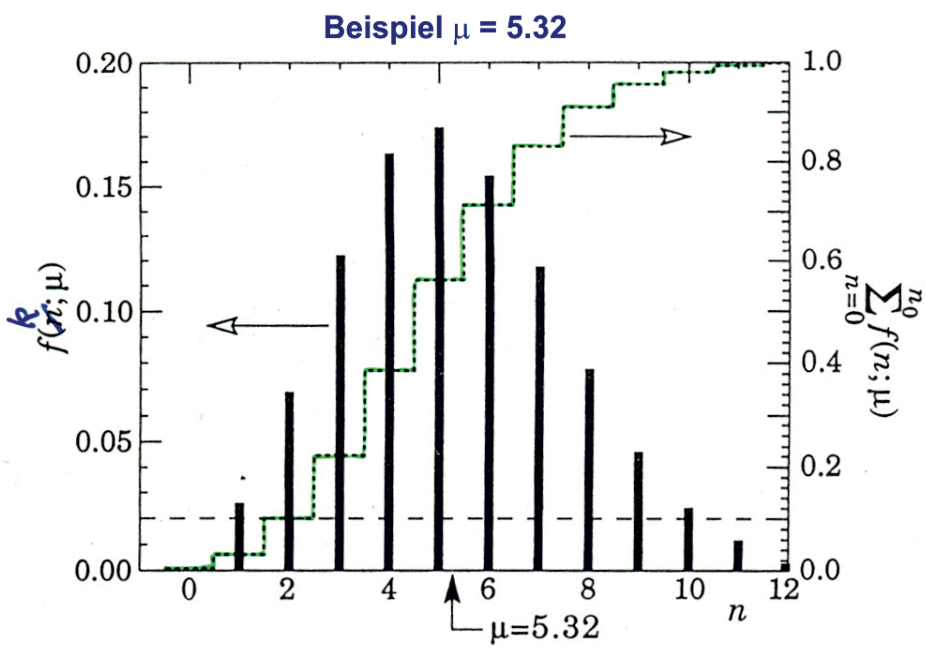


Table 17.3: Poisson upper limits N for n_0 observed events.

Kexp n_0	$\varepsilon =$ 10%	$\varepsilon =$ 5%	n_0	$\varepsilon =$ 10%	$\varepsilon =$ 5%
0	2.30	$\boxed{3.00}$	6	10.53	11.84
1	3.89	4.74	7	11.77	13.15
2	$\boxed{5.32}$	6.30	8	13.00	14.44
3	6.68	7.75	9	14.21	15.71
4	7.99	9.15	10	15.41	16.96
5	9.27	10.51	11	16.60	18.21